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Anomalous response of a stratified medium to volume heat release

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A linear stationary problem of the response of a semi-bounded stably stratified fluid/gaseous medium to volumetric spatially inhomogeneous heat release in the Boussinesq approximation is theoretically investigated. The most important similarity parameters are the analogue of the Rayleigh number and the aspect ratio of the source of buoyancy. For a perturbation in the form of a single horizontal harmonic, an analytical solution has been found that makes it possible to analyze a number of significant regularities. The nontrivial possibility of an intense hydrothermodynamic response to a weak heat release at a certain ratio of the mentioned parameters is found.

Keywords: stratified medium, volume heat release, convection, density flows, analytical solution, intense response, instability.

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Introduction

In this paper, attention is drawn to the substantive possibility of an intense hydrothermodynamic response of a stably stratified liquid/gaseous medium to a weak volumetric heat release.

A rather extensive publications relate to the theory of convection (density flows) over a thermally inhomogeneous horizontal surface in a gravity field (see, for example, [1–5] and the bibliography in these editions). Such problems have, in particular, well-known geophysical applications (for example, local winds in the atmosphere associated with thermal inhomogeneities of the underlying surface). To a lesser extent, similar problems with volumetric sources of heat (buoyancy) were studied. They also have extensive applications. For example, in the same problems of atmosphere dynamics an important role is played by heat sources due to phase transitions of water vapor. Another close example — admixtures that affect the radiation balance of the medium (see, for example, [6–8]). The related mathematical problems are very complex even in the linear approximation (for small perturbation amplitudes) and are usually studied numerically. In this note, we solve a problem that admits a transparent analytical solution. This makes it possible to detect a substantive effect that would be more difficult to notice during numerical simulations.

1. Problem formulation

We consider a semi-bounded stably stratified in temperature (in the atmosphere — by potential temperature [1,2]) medium bounded from below by a horizontal surface. For simplicity, we restrict ourselves to a two-dimensional problem with a volumetric heat source whose intensity Q depends on the horizontal coordinate x and the vertical coordinate z (axis z is directed upwards).

In the absence of the mentioned source (background state), there is a static solution with a constant vertical temperature gradient $\gamma > 0$ (stable background stratification). The presence of a horizontally inhomogeneous source $Q(x, z)$ leads to perturbations of this background state — to the appearance of horizontal thermal inhomogeneities, horizontal variations in the weight of the medium column and the occurrence of flows. The assumed relative smallness of the volumetric heat release amplitude gives the grounds to consider linear perturbations.

The linearized system of equations of hydrothermodynamics for a two-dimensional stationary problem in the Boussinesq approximation has the form [3,4,9,10]:

$$0 = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} + \nu\Delta_2 u, \;\; 0 = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} + \nu\Delta_2 w + g\alpha\theta, \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \;\; \gamma w = \kappa\Delta_2\theta + Q(x, z). \quad (2)$$

Here u, w are the components of the perturbation of the velocity field of the occurring flow along the axes x, z, respectively; p, θ — pressure and temperature perturbations; α — coefficient of thermal expansion, $\bar{\rho}$ — average (reference) density of the medium; g — free fall acceleration; Δ_2 — symbol of the two-dimensional Laplacian; κ, ν — exchange coefficients.

On the lower horizontal boundary (surface $z = 0$) it is assumed that the impermeability and non-sliping conditions are satisfied, as well as fixed temperature (the absence of temperature perturbations):

$$u = w = 0,\ \theta = 0 \text{ at } z = 0. \tag{3}$$

It is assumed that the volumetric heat release intensity is nonzero in the region of finite thickness near the lower boundary. Correspondingly, at $z \to \infty$, perturbation damping is assumed.

2. Solution

Eliminating from the system of equations all unknowns, except for one, the following equation can be easily obtained

$$\Delta_2^3 w + \frac{N^2}{\nu\kappa}\frac{\partial^2 w}{\partial x^2} = \frac{\alpha g}{\nu\kappa}\frac{\partial^2 Q}{\partial x^2}. \tag{4}$$

Here $N = (\alpha g \gamma)^{1/2}$ — buoyancy frequency (Brunt–Väisälä frequency). It is convenient to analyze a model with a harmonic dependence of heat release on the horizontal coordinate:

$$Q(x, z) = q(z)\cos kx. \tag{5}$$

In this case, we also look for the solution in the form of a horizontal harmonic:

$$u(x, z) = U(z)\sin kx,\ \ w(x, z) = W(z)\cos kx,$$

$$\theta(x, z) = \Theta(z)\cos kx,\ \ p(x, z)/\bar{\rho} = P(z)\cos kx.$$

Equation (4) takes the form

$$\left(\frac{d^2}{dZ^2} - 1\right)^3 W - RW = -R\,\frac{q}{\gamma},\ \ R \equiv \frac{N^2}{\kappa\nu k^4}. \tag{6}$$

Here we introduce the dimensionless variable $Z = kz$ and the dimensionless parameter R, which is some analogue of the Rayleigh number [3,10].

We seek the solution of the last equation in the standard way as the sum of the general solution of the homogeneous equation and the particular solution of the inhomogeneous equation. The mentioned general solution can be represented as a linear combination of exponents of the type $\exp(\sigma_j kz)$, where σ_j —the roots of the characteristic equation

$$(\sigma^2 - 1)^3 - R = 0. \tag{7}$$

Taking into account the damping of perturbations at $z \to \infty$, three of the six roots σ_j are selected with negative real parts (here it is assumed that these roots are different):

$$W_h(z) = \sum_{j=1}^{3} C_j \exp(k\sigma_j z), \tag{8}$$

where C_j are constants of integration.

As an example, consider a model with a source of heat (buoyancy) that dampens with height according to an exponential law:

$$q = q_0 \exp(-z/h), \tag{9}$$

where h — some vertical scale, $q_0 > 0$. In this case, it is easy to find a particular solution of the inhomogeneous equation (6): $W_i = W_0 \exp(-z/h)$, where the vertical velocity scale

$$W_0 = \frac{q_0}{\gamma[1 - (1 - \delta^2)^3/(\delta^6 R)]}, \tag{10}$$

dimensionless parameter $\delta = hk$. Taking into account (1) and the continuity equation, the solution can be represented in the form

$$w = \left[\sum_{j=1}^{3} C_j \exp(k\sigma_j z) + W_0 \exp(-z/h)\right]\cos kx,$$

$$u = \left[-\sum_{j=1}^{3} C_j \sigma_j \exp(k\sigma_j z) + (W_0/hk)\exp(-z/h)\right]\sin kx,$$

$$\theta = \frac{\nu k^2}{\alpha g}\left[\sum_{j=1}^{3} C_j(\sigma_j^2 - 1)^2 \exp(k\sigma_j z) + \frac{W_0}{\delta^4}(1 - \delta^2)^2 \exp(-z/h)\right]\cos kx. \tag{11}$$

Taking into account the boundary conditions (3), we obtain the system of equations for determining the integration constants C_j:

$$\sum_{j=1}^{3} C_j = -W_0,\ \ \sum_{j=1}^{3} \sigma_j C_j = W_0/\delta,$$

$$\sum_{j=1}^{3} (\sigma_j^2 - 1)^2 C_j = -\frac{W_0}{\delta^4}(1 - \delta^2)^2. \tag{12}$$

As can be seen from (7), the value $\sigma_j^2 - 1$ can take three values: $R^{1/3}$, $R^{1/3}\exp(\pm 2\pi i/3)$. The expressions for the roots σ_j are somewhat cumbersome in the general case. It makes sense to use the limiting case of large values of the parameter R. For example, if in the surface layer of the atmosphere $N = 10^{-2}$ s, $K = 3\,\mathrm{m^2/s}$ (rather characteristic values), then for $k = 10^{-2}\,\mathrm{m^{-1}}$ (half-wavelength about 300 m) $R = 10^3$. In the specified limit $|\sigma_j| \gg 1$, the values of roots are with negative real parts:

$$\sigma_1 \approx -R^{1/6},$$

$$\sigma_2 \approx -R^{1/6}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = R^{1/6}\exp(-2\pi i/3),$$

$$\sigma_3 \approx -R^{1/6}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = R^{1/6}\exp(2\pi i/3). \tag{13}$$

The approximate solution of the system (12) has the form:

$$C_1 \approx \frac{1}{2}(W_1 - W_2),$$

$$C_2 \approx \frac{1}{\sqrt{3}}\left[W_0 \exp\left(-\frac{5}{6}\pi i\right) + \frac{1}{2} W_1 \exp\left(\frac{5}{6}\pi i\right) + \frac{\sqrt{3}}{2} W_2 \exp\left(\frac{\pi i}{3}\right)\right],$$

$$C_3 \approx \frac{1}{\sqrt{3}}\left[W_0 \exp\left(\frac{5}{6}\pi i\right) + \frac{1}{2} W_1 \exp\left(-\frac{5}{6}\pi i\right) + \frac{\sqrt{3}}{2} W_2 \exp\left(-\frac{\pi i}{3}\right)\right]. \quad (14)$$

Here the velocity scales are introduced

$$W_1 = -\frac{W_0}{b^4}(1-\delta^2)^2, \;\; W_2 = W_0/b; \quad (15)$$

dimensionless parameter $b = \delta R^{1/6}$. Below it will also be convenient to use the vertical scale $H = 1/kR^{1/6}$. Explicit form of the solution taking into account (14)

$$w = \left\{\frac{1}{2}(W_1 - W_2)\exp\left(-\frac{z}{H}\right) + \frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}\frac{z}{H}\right) \times \left[W_0 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} + \frac{5}{6}\pi\right) + \frac{1}{2} W_1 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{5}{6}\pi\right) + \frac{\sqrt{3}}{2} W_2 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{\pi}{3}\right)\right] + W_0 \exp\left(-\frac{z}{h}\right)\right\}\cos kx,$$

$$u = R^{1/6}\left\{\frac{1}{2}(W_1 - W_2)\exp\left(-\frac{z}{H}\right) - \frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}\frac{z}{H}\right) \times \left[W_0 \sin\left(\frac{\sqrt{3}}{2}\frac{z}{H}\right) + \frac{1}{2} W_1 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} W_2 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} + \frac{\pi}{3}\right)\right] + \frac{W_0}{b}\exp\left(-\frac{z}{h}\right)\right\}\sin kx,$$

$$\theta = \frac{\nu k^2}{\alpha g} R^{2/3}\left\{\frac{1}{2}(W_1 - W_2)\exp\left(-\frac{z}{H}\right) + \frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}\frac{z}{H}\right) \times \left[W_0 \sin\left(\frac{\sqrt{3}}{2}\frac{z}{H}\right) + \frac{1}{2} W_1 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} W_2 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} + \frac{\pi}{3}\right)\right] - W_1 \exp\left(-\frac{z}{h}\right)\right\}\cos kx,$$

$$\frac{p}{\rho} = k\nu R^{1/2}\left\{-\frac{1}{2}(W_1 - W_2)\exp\left(-\frac{z}{H}\right) + \frac{2}{\sqrt{3}} \times \exp\left(-\frac{1}{2}\frac{z}{H}\right)\left[W_0 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} + \frac{5}{6}\pi\right) + \frac{1}{2} W_1 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{5}{6}\pi\right) + \frac{\sqrt{3}}{2} W_2 \cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{\pi}{3}\right)\right] - \frac{W_0}{R^{1/2}\delta^3}(1-\delta^2)\exp\left(-\frac{z}{h}\right)\right\}\cos kx.$$

3. Solution analysis

For certainty we consider the solution near the vertical $x = 0$. In this region, the heat source is $Q > 0$. Therefore, one can expect generally positive temperature deviations, negative density deviations, a decrease in pressure (a decrease in the weight of the medium column), converging horizontal flows and (due to continuity) upward movements. This is what Fig. 1 prepared according to the obtained solution demonstrates. Two thin curves correspond to a set of parameter values typical for the surface layer of atmosphere: $q_0 = 3 \cdot 10^{-4}$ K/s (about 1 K/hour), $\kappa = \nu = 3\,\mathrm{m}^2/\mathrm{s}$, $h = 50$ m, $k = 2 \cdot 10^{-3}\,\mathrm{m}^{-1}$, $\gamma = 3 \cdot 10^{-3}$ K/m. Moreover, $R \approx 7 \cdot 10^5$, $\delta = 0.1$, $H \approx 50$ m.

At sufficiently large spatial scales of the source (in particular, large values of the parameter R) and far from the lower boundary, the diffusion terms in the equations are insignificant, and, as can be seen already from (4), there is an approximate solution

$$w \approx Q/\gamma. \quad (16)$$

Its physical meaning is quite clear: each heated element of the medium moves vertically at such velocity that the increase in buoyancy due to heat release is approximately compensated by a decrease in buoyancy due to ascent into less dense layers of the medium. The temperature and pressure deviations are small in this case, so this solution can be called a neutral buoyancy mode. But near the lower boundary, due to the impermeability condition, the vertical velocity is small, and the upward motions are not able to carry away all the released heat. Therefore, the buoyancy deviations there are relatively large, and diffusion terms are significant.

It is interesting to note that here, in principle, there is the possibility of an unusually intense response to heat

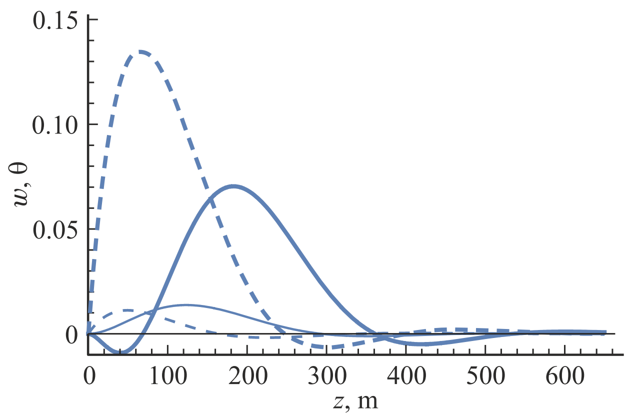

Figure 1. Examples of vertical profiles of temperature perturbations (solid lines, normalized to $2\nu k^2 R^{2/3}/\alpha g$) and vertical velocity (dashed lines, [m/s]) on the axis of the heat release region (on the vertical $x = 0$). The thin lines correspond to the situation in general position ($h = 50$ m). Thick lines ($h \approx 52.82$ m) correspond to the range of parameter values near the relationship $\delta = (1 + R^{1/3})^{-1/2}$, when the denominator (10) close to zero.

release. The denominator of the expression (10) can, generally speaking, pass through zero. This happens with the following (quite actual) relationship of dimensionless parameters:

$$\delta = (1 + R^{1/3})^{-1/2}. \quad (17)$$

For $R \gg 1$ this corresponds to close values of the vertical scale h of the source and the vertical scale H due to the stable stratification of the medium. The values of the velocity scales W_0, W_1, W_2 when executing (17) obviously turn to infinity. The linear model considered in this paper, of course, allows us to consider only not too large perturbation amplitudes. Fig. 1 (thick lines) demonstrates that as the values of the parameters approach (17) (compared to the accepted above parameter values, the scale h is only slightly changed), the amplitude of temperature and velocity perturbations sharply increases (increases by several times in comparison with thin lines corresponding to typical amplitudes far from relationship (17)). Note that the discovered effect is very sensitive to the values of the parameters near relationship (17).

The possibility of an intense response to weak heat release means the existence of a linear instability of the considered background state with respect to perturbations of a certain structure. The instability of stably stratified (heated from above) shearless viscous medium looks, at first glance, paradoxical. But for a two-layer system heated from above, similar possibilities were theoretically discovered earlier [9,11]; even the term „anticonvection"appeared [11]. This non-trivial effect is schematically illustrated in Fig. 2. Let the heat source (shaded in the figure) be distributed near the interface between two media (the lower medium is assumed to be much denser than the upper one, so that the interface deformations are insignificant). In both media near the region of heat release the convective flows arise, which carry away the released heat. (We emphasize that since the heat release against the background of sufficiently stable stratification is assumed to be weak, we are not talking about the occurrence of convective instability. Flows arise for a different reason: due to the horizontal inhomogeneity of the hydrostatic pressure — the occurrence of horizontal gradients of the weight of the medium column. Using another terminology, these flows can be called density flows). As can be seen from the Figure, near the interface the occurring horizontal flows are directed towards each other. Rigorous calculations based on a linear model show that under certain ratios of media parameters, the interaction of these two counterflows can lead to violation of „natural ventilation" of the heat release area. For example, a converging horizontal flow over the interface can, due to viscosity, also entrain the lower medium in the same direction. The latter thus stops to carry heat out of the region of heat release, so that when the interaction of the media is taken into account, the direction of flow in the lower medium shown in Fig. 2 changes, heat accumulates, and the perturbations increase.

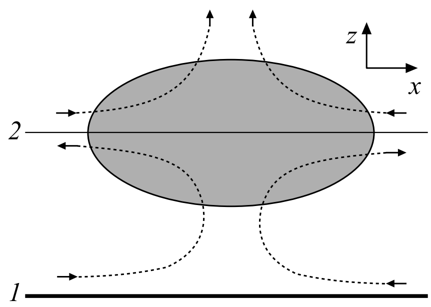

Figure 2. Scheme of flows arising near the heat release region (shaded) in a two-layer medium: *1* — lower boundary of the medium, *2* — interface between two media, dotted lines — streamlines.

Until now, it was believed that such amplification of perturbations is possible only in certain two-layer media with very special relations between the parameters of two media [11]. The above calculations show for the first time that a similar effect is, in principle, also possible in a single-layer semi-bounded continuously stratified medium. In such medium, some analogue of the horizontal interface is the level $z \sim H = 1/kR^{1/6}$. The region of heat release located below this level leads to the appearance of flows that are qualitatively close to the flows in the lower medium in Fig. 2. Indeed, in the paper [4] it is shown that perturbations from the source concentrated on the lower boundary reach a height of about H (movements cannot penetrate above because of stable stratification). The flow pattern obtained in [4] is qualitatively close to the flow pattern below the interface in Fig. 2. It is obvious that the heat release in the region $0 < z \leq H$ also cannot induce perturbations penetrating much higher than $z = H$. At higher levels, solution (16) is approximately satisfied, it is qualitatively close to the structure of perturbations above the interface in Fig. 2. Thus, a physical mechanism is seen that is similar to the previously studied situation in a two-layer medium.

Conclusion

As mentioned above, convection over a thermally inhomogeneous horizontal surface was studied in detail earlier in the literature. It was shown that perturbations penetrate from the lower boundary into a stably stratified medium up to a height of about $H \approx (\nu\kappa L^2/N^2)^{1/6}$, where L — horizontal scale of thermal inhomogeneity (see, for example, [4]). If the source of buoyancy is volumetric, but concentrated relatively close to the lower boundary (in the region $z < H$), then it could be expected that the result will not change qualitatively, and this indeed follows from the obtained solution. But, as can be seen from the solution, the results change qualitatively when the vertical scales of the source of buoyancy reach and exceed the value H. Far from the lower boundary, a mode with neutral buoyancy arises, in which $w \approx Q/\gamma$. And at parameter values close to condition (17) despite stable background stratification the intense hydrothermodynamic response to weak heat release is possible. Note that this is a rigorously proven result, since it can already be seen from the easily reproduced and analyzed formula (10), where the denominator can be zero. The discovered possibility of intense response to weak heat release means the presence of the linear instability of stably stratified medium with respect to perturbations of a certain structure. Previously, this possibility was shown only for some two-layer media.

Conflict of interest

The author declares that he has no conflict of interest.

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