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The possibility of quantitative determination of the boundary of generalized synchronization using nearest neighbor and phase tube methods

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The possibility of quantitative determination of the generalized synchronization boundary in two mutually coupled systems with different attractor topology using the nearest neighbor and phase tube methods has been established. The obtained results have been compared with the results of calculating the spectrum of Lyapunov exponents for interacting systems. Estimation of the accuracy of determining the generalized synchronization boundary in comparison with known methods and approaches has been made. The obtained results have been illustrated using the examples of Ressler systems, Lorenz oscillators, Chua and Kiyashko–Pikovsky–Rabinovich generators.

Keywords: generalized synchronization, mutually coupled systems, spectrum of Lyapunov exponents, nearest neighbor method, phase tube method.

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Generalized synchronization is one of the intriguing types of synchronous behavior that are currently being examined actively [1–4]. This regime has first been discovered in unidirectionally coupled systems [5], but the concept of generalized synchronization was extended later to mutually coupled systems and networks of coupled nonlinear elements [6]. In all these cases, generalized synchronization is understood as the establishment of a functional between the states of interacting systems [7]. The method of calculation of the spectrum of Lyapunov exponents [8], the auxiliary system approach [9], the nearest neighbor method [5], and the phase tube method [7] are used as diagnostic techniques for this regime.

Each of the above methods has its own advantages and drawbacks. For example, the auxiliary system approach provides an opportunity to identify relatively accurately the onset of generalized synchronization in unidirectionally coupled systems with an explicitly stated evolution operator and is used fairly often to determine the characteristics of intermittent behavior observed near the boundary of this regime [2,10]. At the same time, it is inapplicable in the examination of this regime of systems with mutual coupling [11].

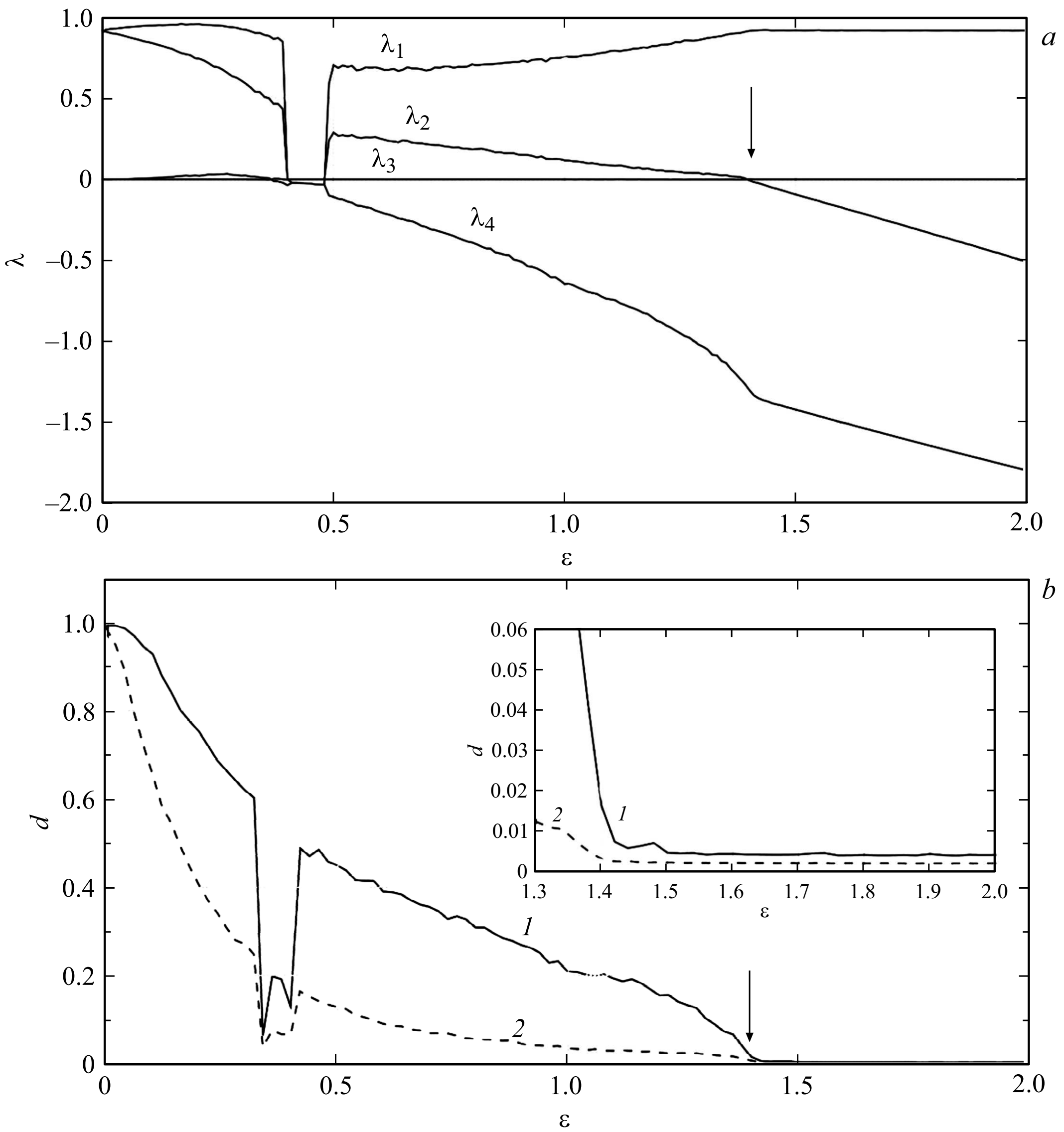
The method of calculation of the spectrum of Lyapunov exponents is equally efficient in both unidirectionally and mutually coupled systems, but only if the equations characterizing the dynamics of interacting systems are given explicitly. If the evolution operator is unknown (e.g., when experimental time series are analyzed), it turns out to be problematic to either estimate the Lyapunov exponents governing the introduction of generalized synchronization [12] or produce an identical copy of a time series corresponding to an auxiliary system to perform diagnostics of generalized synchronization using the auxiliary system approach [13].

The other two mentioned techniques (nearest neighbor and phase tube methods) are, on the contrary, approximate, but allow one to identify fairly easily the presence of generalized synchronization by examining experimental time series without any regard to the type of coupling between interacting systems. The common procedure for identifying the presence of generalized synchronization by the nearest neighbor method involves setting several reference states in the phase space of the first system and finding their nearest neighbors and images in the phase space of the second system [5,6]. If images of the nearest neighbors are distributed throughout the whole attractor of the second system, generalized synchronization between interacting systems is lacking. If, in contrast, they are confined to certain regions of the attractor corresponding to the positioning of nearest neighbors themselves, generalized synchronization is established.

The following parameter is traditionally used as a quantitative estimate of the proximity of states of interacting systems [14]:

$$d = \frac{1}{N\delta}\sum_{k=0}^{N-1} \| u_k - u_{kn} \|, \qquad (1)$$

where N is the number of reference states, δ is the mean distance between states of one of the examined systems, x_k are reference states of this system, x_{kn} are their nearest neighbors, and u_k and u_{kn} are the images of x_k and x_{kn} in the phase space of the other system, respectively. This parameter is close to unity if systems are interacting in the



Dependences of four largest Lyapunov exponents (a) and quantitative measure d (b) on coupling parameter ε for two mutually coupled Lorenz systems. Curves *1* and *2* in the lower panel represent the results of application of the nearest neighbor method and the phase tube method (phase tube length $T = 25$), respectively. The generalized synchronization threshold is marked by an arrow in both panels. An enlarged view of the threshold region is presented in the inset in the lower panel.

asynchronous regime and assumes near-zero values in the regime of generalized synchronization.

The nearest neighbor method differs from the phase tube one only in the procedure of searching for nearest neighbors. While neighbors residing at a distance shorter than a given length at just a specific moment of time are regarded as the nearest ones in the first case, the phase tube method stipulates that only the states remaining close throughout a certain interval of time, which is called the prehistory or the phase tube length, may be regarded as close ones. The phase tube method has a fundamental advantage over the nearest neighbor method in providing an opportunity to determine precisely the presence of a functional between states of interacting systems [7] and in retaining its efficiency in systems with a complex (two-sheeted) attractor topology, where the nearest neighbor method yields erroneous results of identification of generalized synchronization [15].

A quantitative estimate of the proximity of states of interacting systems obtained using the phase tube method is often left unaddressed in literature. At the same time, it is evident that the same criterion (calculation of parameter d specified by Eq. (1)) may be used in the diagnostics of generalized synchronization with the phase tube method, with the sole difference being that, as was noted above, the algorithm for searching for nearest neighbors should be altered.

Importantly, the nearest neighbor and phase tube methods are similar in many respects to the recurrence-based approach (see, e.g., [16,17]). The latter method finds application in the analysis of experimental time series of various

Table 1. Equations and control parameter values of the studied systems

Studied system	Equations	Parameter values
Ressler systems	$\dot{x}_{1,2} = \omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2})$ $\dot{y}_{1,2} = \omega_{1,2}x_{1,2} + ay_{1,2}$ $\dot{z}_{1,2} = b + z_{1,2}(x_{1,2} - c)$	$a = 0.15$, $b = 0.2$, $c = 10$, $\omega_1 = 1.00$ $\omega_2 = 0.98$
Ressler systems	$\dot{x}_{1,2} = \sigma(y_{1,2} - x_{1,2})$ $\dot{y}_{1,2} = x_{1,2}(r_{1,2} - z_{1,2}) - y_{1,2} + \varepsilon(y_{2,1} - y_{1,2})$ $\dot{z}_{1,2} = x_{1,2}y_{1,2} - bz_{1,2}$	$\sigma = 10$, $r_1 = 28.0$, $r_2 = 28.1$, $b = 8/3$
Chua generators	$\dot{x}_{1,2} = c_1\big(y_{1,2} - x_{1,2} - g(x_{1,2})\big)$ $\dot{y}_{1,2} = c_2(x_{1,2} - y_{1,2} + z_{1,2}) + \varepsilon(y_{2,1} - y_{1,2})$ $\dot{z}_{1,2} = -c_3y_{1,2}$	$c_1 = 15.6$, $c_2 = 1.0$, $c_3 = 25.580$, $g(x) = m_1x + \frac{m_0-m_1}{2}(\lvert x+1\rvert - \lvert x-1\rvert)$
Kiyashko–Pikovsky–Rabinovich generators	$\dot{x}_{1,2} = \omega_{1,2}\big(h\big(x_{1,2} - \varepsilon(y_{2,1} - y_{1,2})\big) + y_{1,2} - z_{1,2}\big)$ $\dot{y}_{1,2} = -x_{1,2} + \varepsilon(y_{2,1} - y_{1,2}\big)$ $\dot{z}_{1,2} = \big(x_{1,2} - f(z_{1,2})\big)/m$	$h = 0.2$, $m = 0.1$, $\omega_1 = 1.07$, $\omega_2 = 1.04$, $f(x) = -x + 0.002\sinh(5x - 7.5) + 2.9$

Table 2. Boundaries of generalized synchronization determined for the examined systems by calculating the spectrum of Lyapunov exponents, the nearest neighbor method, and the phase tube method

Studied system	Method of calculation of the spectrum of Lyapunov exponents	Nearest neighbor method		Phase tube method	
		Boundary	Accuracy	Boundary	Accuracy
Ressler systems	0.104	0.116	0.115	0.116	0.115
Lorenz systems	1.4	1.44	0.029	1.4	0
Chua generators	1.04	0.92	0.115	0.9	0.135
Kiyashko–Pikovsky–Rabinovich generators	0.06	0.088	0.467	0.08	0.333

nature (including such analysis that is aimed at identifying different types of chaotic synchronization). In order to check for the presence of generalized synchronization, one needs to construct recurrence plots for each interacting system and a joint recurrence plot for both systems for every value of the system coupling parameter. The issue of determination of the threshold of generalized synchronization with this method is often left undiscussed in literature.

The present study is the first attempt at quantitative determination of the threshold of generalized synchronization in mutually coupled systems with different attractor topologies via the nearest neighbor and phase tube methods. Ressler and Lorenz systems and Chua and Kiyashko–Pikovsky–Rabinovich generators are examined as examples of such systems. The equations and control parameter values of the studied systems are listed in Table 1. The control parameter values were chosen so that all interacting systems remained in the chaotic regime: band chaos was observed in Ressler systems and Kiyashko–Pikovsky–Rabinovich generators, while the attractors in Lorenz systems and Chua generators had a two-sheeted structure. In addition to the nearest neighbor and phase tube methods, the method of calculation of the spectrum of Lyapunov exponents was also used to identify generalized synchronization.

Dependences of four largest Lyapunov exponents and quantitative measure d on coupling parameter ε for two mutually coupled Lorenz systems are presented in the figure for illustrative purposes. It is evident that the second-largest Lyapunov exponent enters the region of negative values at $\varepsilon = 1.4$, which corresponds to the introduction of generalized synchronization into the examined system. Dependences $d(\varepsilon)$ calculated using the nearest neighbor and phase tube methods reach saturation at approximately the same coupling parameter value. The saturation level is sufficiently high in the case of the nearest neighbor method and near-zero for the phase tube method.

Similar results were obtained for other systems. To illustrate this, the boundaries of generalized synchronization calculated for all four systems mentioned above with the use of three examined methods are listed in Table 2. It can be seen that, although the numerical values of synchronization thresholds differ slightly, the differences are insignificant (especially so in the case of the phase tube method). Therefore, these approaches may be applied in both qualitative and quantitative determination of the boundary of generalized synchronization.

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Conflict of interest

The authors declare that they have no conflict of interest.

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