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Radiation in the graphene: kinetic approach

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The work briefly summarized the latest results of the kinetic theory of radiation in the graphene, consisting of both the quasiclassical component generated by plasma currents and the quantum component generated by direct interaction with carriers, the excitement of which is described by nonperturbative methods. The achieved level of development of theory allows us to talk about a qualitative level of consent with existing experimental data.

Keywords: nonperturbative kinetic, strong fields, graphene, quasiclassical and quantum radiation.

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Introduction

It is well known that graphene at the low-energy limit is described by $D = 2 + 1$ massless two-velocity quantum field model [1,2] with a non-analytical dependence on the strength of the external electric field already in the region of weak fields (e.g. [3]). This results in the experimental accessibility in the study of electron-hole plasma (EHP) generation in external fields (Landau–Zener effect [4,5]), the equivalent of which in the strong field QED is the Sauter–Schwinger effect [6,7]. A good agreement has been reached with the experiment at this level [8,9]. Problems occur at the stage of matching $D = 2 + 1$ dynamics of graphene with $D = 3 + 1$ dynamics of the interaction of internal currents with a plasma quasiclassical field and with a quantum field generated by direct interaction with carriers. The paper [10] outlined a solution for this problem by semi-phenomenological $D = 3 + 1$ modification of QED in graphene.

We implement this approach in this paper using the nonperturbative kinetic theory both at the level of a self-consistent description of interaction with a plasma field (problem of back-reaction (sec. 2)), and at the level of the interaction of the EHP with a quantized field (section 3). A similar problem has to be considered when describing radiation (quasiclassical and quantum) into the outer regions of space with respect to the graphene plane. The results of the study are briefly discussed in the Conclusion.

1. Kinetic equation

The basic kinetic equation (KE) in graphene in the approximation of a self-consistent field was obtained in a nonperturbative basis in Ref. [11] by the analogy with electron-positron plasma in a strong field QED (for example, [12]).

These studies assume that the effective electric field with a vector potential $A^{(k)}(t)$ (indices $k = 1, 2$ correspond to two spatial dimensions of three-dimensional Euclidean space) and a field strength $E^{(k)}(t) = -(1/c)\dot{A}^{(k)}(t)$ is spatially homogeneous and time-dependent. In general, it consists of external and internal (plasma) fields,

$$A^{(k)}(t) = A^{(k)}_{ex}(t) + A^{(k)}_{in}(t). \tag{1}$$

It is sufficient to use the approximation of the external field $E(t) = E_{ex}(t)$ for evaluation of the effectiveness of the EHP creation, while a self-consistent description of plasma oscillations requires the use of a full field.

The basic KE describes excitations in graphene in terms of quasiparticles with quasienergy $\varepsilon(\mathbf{p}, t) = v_F\sqrt{P^2}$ ($v_F = 10^6$ m/s — Fermi velocity, $\mathbf{p} = (p^{(1)}, p^{(2)}, 0)$ — quasiparticle momentum) and quasimomentum $P^{(k)} = p^{(k)} - (e/c)A^{(k)}(t)$ using the distribution function $f(\mathbf{p}, t)$. This KE takes into account the condition of electroneutrality of the electron and hole subsystems $f(\mathbf{p}, t) = f_e(\mathbf{p}, t) = f_h(-\mathbf{p}, t)$ and can be written either in the form of non-Markov type integro-differential equation [11]:

$$\dot{f}(\mathbf{p}, t) = \frac{1}{2}\lambda(\mathbf{p}, t)\int_{t_0}^{t} dt'\lambda(\mathbf{p}, t')[1-2f(\mathbf{p}, t')]\cos\theta(\mathbf{p}; t, t') \tag{2}$$

or in the equivalent form of a system of ordinary differential equations

$$\dot{f}(\mathbf{p}, t) = \frac{1}{2}\lambda(\mathbf{p}, t)u(\mathbf{p}, t),$$

$$\dot{u}(\mathbf{p}, t) = \lambda(\mathbf{p}, t)[1 - 2f(\mathbf{p}, t)] - \frac{2\varepsilon(\mathbf{p}, t)}{\hbar}v(\mathbf{p}, t),$$

$$\dot{v}(\mathbf{p}, t) = \frac{2\varepsilon(\mathbf{p}, t)}{\hbar}u(\mathbf{p}, t), \tag{3}$$

where

$$\lambda(\mathbf{p}, t) = \frac{ev_F^2[E^{(1)}(t)P^{(2)} - E^{(2)}(t)P^{(1)}]}{\varepsilon^2(\mathbf{p}, t)}, \tag{4}$$

$$\theta(\mathbf{p};t,t') = \frac{2}{\hbar}\int\limits_{t'}^{t} d\tau\,\varepsilon(\mathbf{p},\tau). \tag{5}$$

It is assumed that the external field is turned on at time $t = t_0$. The paper [11] provides the procedure of the transition from KE (2) to the system of equations (3). Some properties of KE in the form (2) and (3) and their solutions for various field models are discussed in papers [11,13].

The distribution function $f(\mathbf{p},t)$ allows calculating the average values of physical quantities. For example, the density of the quasiparticles and the energy density are equal to

$$n(t) = 2N_f \int [dp] f(\mathbf{p},t), \tag{6}$$

$$\mathscr{E}_{eh}(t) = 2N_f \int [dp]\varepsilon(\mathbf{p},t) f(\mathbf{p},t), \tag{7}$$

where $[dp] = d^2p(2\pi\hbar)^{-2}$ and $N_f = 4$ — the number of different quasiparticle flavors (two Dirac points and two pseudospin states), factors 2 in (6) and (7) are attributable to the equality of contributions of electrons and holes. The total energy density of quasiparticle excitations

$$\mathscr{E}(t) = \mathscr{E}_{eh}(t) + \mathscr{E}_{pol}(t) \tag{8}$$

also includes the polarization energy density [11]

$$\mathscr{E}_{pol}(t) = -N_f\hbar \int [dp]\lambda(\mathbf{p},t)v(\mathbf{p},t). \tag{9}$$

The total current density $j^{(k)}(t)$ consists of conductive and polarization currents [11],

$$j^{(k)}(t) = j^{(k)}_{cond}(t) + j^{(k)}_{pol}(t), \tag{10}$$

$$j^{(k)}_{cond}(t) = 2N_f e \int [dp] v^{(k)}_g(\mathbf{p},t) f(\mathbf{p},t), \tag{11}$$

$$j^{(k)}_{pol}(t) = -N_f e \int [dp] v^{(k)}_{pol}(\mathbf{p},t) u(\mathbf{p},t), \tag{12}$$

where

$$v^{(k)}_g(\mathbf{p},t) = \frac{\partial\varepsilon(\mathbf{p},t)}{\partial p^{(k)}} = \frac{v_F^2 P^{(k)}}{\varepsilon(\mathbf{p},t)}, \tag{13}$$

$$v^{(k)}_{pol}(\mathbf{p},t) = \varepsilon(\mathbf{p},t)\frac{\partial\lambda(\mathbf{p},t)}{\partial E^{(k)}(t)} = \frac{v_F^2}{\varepsilon(\mathbf{p},t)}\begin{cases} P^{(2)}, & k=1,\\ -P^{(1)} & k=2,\end{cases} \tag{14}$$

wherein $\mathbf{v}_g\mathbf{v}_{pol} = 0$. The currents (11), (12) are determined by the effective field (1). From formulas (12) and (9) it follows that the auxiliary functions $u(\mathbf{p},t)$ and $v(\mathbf{p},t)$ in system (3) describe polarization effects in current and energy densities, respectively. The sign on the right side of the polarization current (12) determines the dampening of the conductive current (11) (see below).

2. Back-reaction problem

The next level of description takes into account the back-reaction (BR) associated with the generation of internal plasma currents and fields $E^{(k)}_{in}(t) = -(1/c)\dot{A}^{(k)}_{in}$ $(k = 1, 2)$. The problem occurs at the stage of formulation of the Maxwell equation, where the electric field strength $E^{(k)}_{in}$ is compared with the plasma current density. Some dimensional modification has to be introduced in the graphene dynamics due to the assumption of the fundamental nature of the standard $D = 3 + 1$ QED.

Such modifications were proposed in Ref. [10]. The wave function of carriers $\psi(\mathbf{x},t)$ $\left(\mathbf{x} = (x^{(1)}, x^{(2)})\right)$ of $D = 2 + 1$ dimension was modified to the case of $D = 3 + 1$ space-time using the definition as

$$\psi(\mathbf{x},z;t) = \psi(\mathbf{x},t)\frac{1}{\sqrt{d}}\varphi(z)e^{ip^{(3)}z/\hbar}, \quad z = x^{(3)}, \tag{15}$$

where $d \simeq 10^{-8}$ cm — the graphene layer thickness. The dimensionless function $\varphi(z)$ describes the distribution of carriers in the transverse direction and satisfies the conditions of normalization and confinement $(\varphi(z = 0, d) = 0)$. The details of this distribution are further ignored, so that $\varphi(z)e^{ip^{(3)}z/\hbar} \to 1$, which leads to $D = 3 + 1$ modified wave function

$$\tilde{\psi}(\mathbf{x},t) = d^{-1/2}\psi(\mathbf{x},t). \tag{16}$$

This result leads to the following rule for dimensional modification of average quantities of type a:

$$\langle\tilde{a}\rangle = d^{-1}\langle a\rangle, \tag{17}$$

where $\langle a\rangle$ corresponds to $D = 2 + 1$ theory.

Now the Maxwell equation for back-reaction in graphene can be written:

$$\dot{E}^{(k)}_{in}(t) = -4\pi\tilde{j}^{(k)}(t) = -4\pi d^{-1}j^{(k)}(t), \tag{18}$$

where $D = 2 + 1$ densities of conductive and polarization currents are determined by the equations (10)–(12).

The system of kinetic equations (2)(or (3)) and the Maxwell equation (18) describes a self-consistent evolution of the EHP and the internal field. Two stages of the back-reaction processes can be identified here: the excitation of the EHP, which is limited by the period of action of the external field, and the period of free evolution of the system, which has the form of periodic self-consistent plasma oscillations [11]. We study below the back-reaction of graphene in the model of a field of a single Gaussian pulse.

$$E_{ex}(t) = E_0 e^{-t^2/2\tau^2}. \tag{19}$$

In the case of electron-positron and parton plasmas, the mechanism of back-reaction has been well studied on the basis of nonperturbative kinetic theory in a large number of papers, starting with [14]. The masslessness of the theory is the principal feature of graphene. This results in an almost inertialess response of the system and a sharp

weakening of the memory effect in the kinetic description. The internal plasma field strongly grows as a result and almost offsets the external field, so that the effective field is strongly dampened. This results in the suppression of the EHP generation.

These features are very well manifested in case of numerical study of the problem of back-reaction in graphene. The strengths of electric fields (external (E_{ex}), internal (E_{in}) and effective (E)) are shown in Fig. 1, 2: the total field $E = E_{ex} + E_{in}$ is strongly dampened. Plasma oscillations appear after switching of the external field, but they are weakly expressed.

This results in the depletion of EHP. The depletion coefficient for particle densities in the out-state and the current amplitude depletion coefficient is introduced to

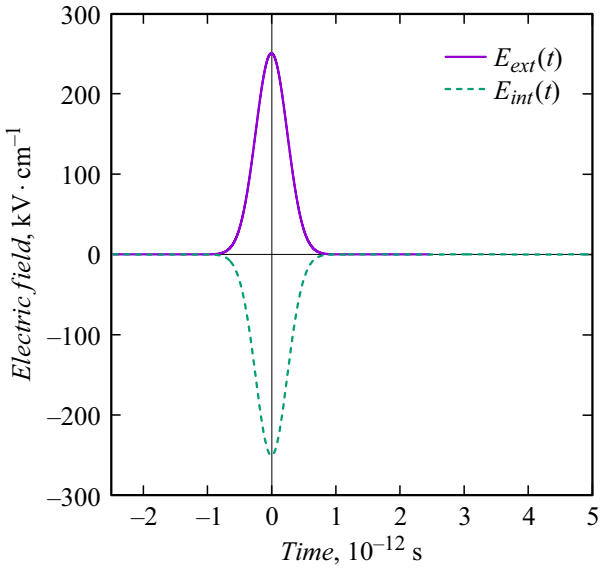

Figure 1. Evolution of electric fields in graphene.

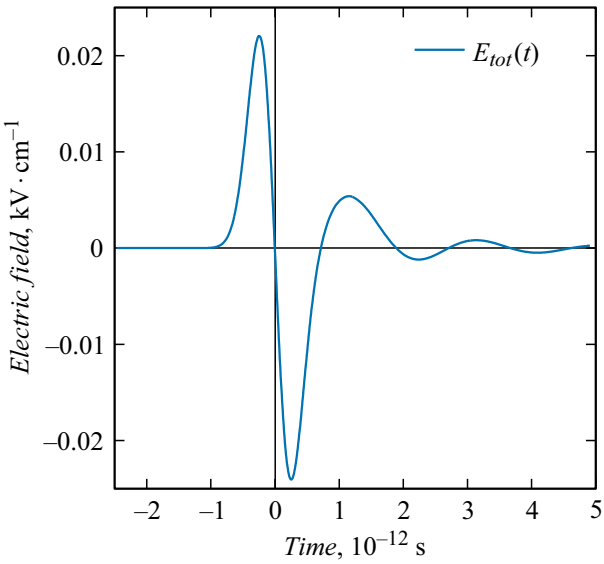

Figure 2. Evolution of the resulting field in graphene.

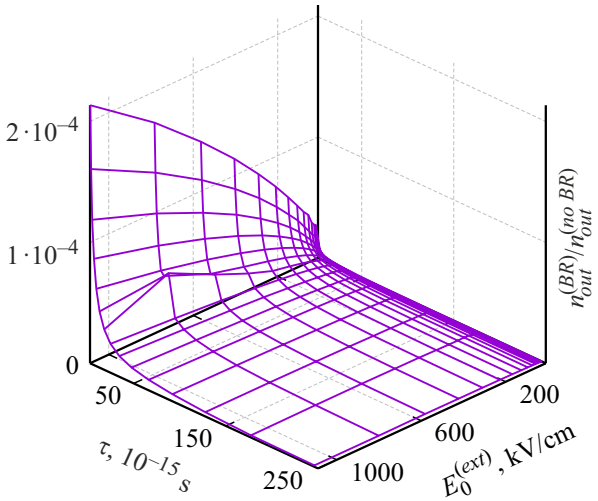

Figure 3. Particle density depletion coefficient.

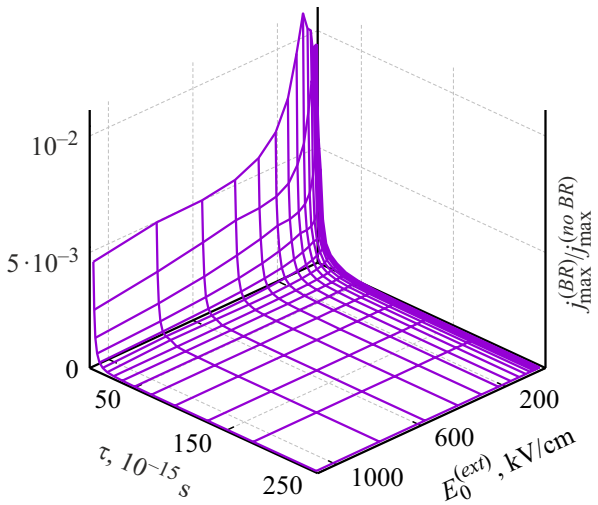

Figure 4. Current density depletion coefficient.

characterize the efficiency of this process:

$$\xi^{(n)} = \frac{n_{BR}^{out}}{n^{out}}, \quad \xi^{(j)} = \frac{j_{BR}^{\max}}{j^{\max}}, \tag{20}$$

where n_{BR}^{out} and $j_{BR}^{\max}$ are calculated taking into account back-reaction, and back-reaction is not taken into account in n^{out} and $j^{\max}$ (Fig. 3, 4). These figures show that both depletion coefficients (20) are very small with a wide variation of the parameters of the external field (19). The conclusion about the depletion of EHP due to the back-reaction is in qualitative agreement with the results of the semi-phenomenological theory of cascade processes in electron-positron plasma (review [15]), confirming N. Bohr's idea that a critical field strength $E_c = m^2/e$ cannot be achieved in fields capable of generating electron-positron plasma.

E_0- and τ-dependencies can be restored at a qualitative level in the definitions of (20) densities:

$$n^{out} \propto E_0^{3/2}\tau, \quad n_{BR}^{out} \propto E_0^2, \tag{21}$$

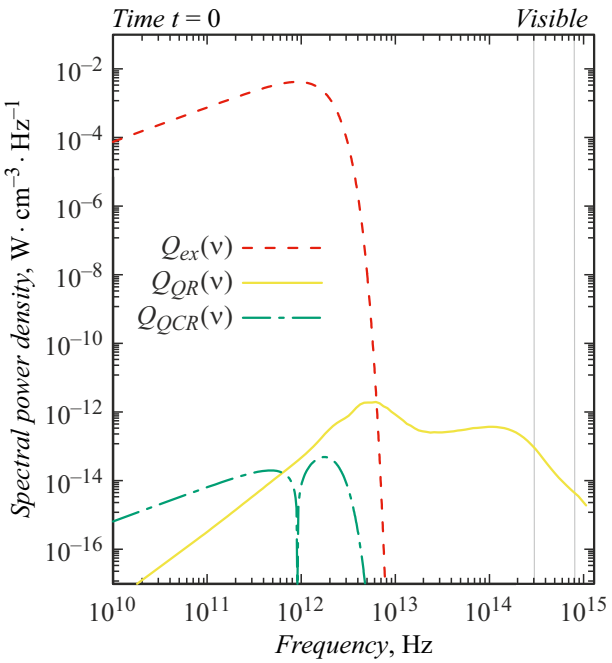

Figure 5. Spectral power densities of radiation with $E_0 = 250\,\text{kV/cm}$, $\tau = 2.46 \cdot 10^{-13}$ c.

$$j^{\max} \propto E_0^{3/2}\tau, \quad j_{BR}^{\max} \propto E_0\tau^{-1}. \tag{22}$$

These estimates lead to the following results for the depletion coefficients

$$\xi^{(n)} \propto E_0^{1/2}\tau^{-1}, \quad \xi^{(j)} \propto E_0^{-1/2}\tau^{-2}. \tag{23}$$

3. Quasi-classical and quantum radiation

The quasiclassical radiation (QCR) in the outer regions relative to the graphene plane is determined by internal plasma currents. The energy density of this radiation at a great distance from the graphene plane is (see Appendix)

$$\mathscr{E}_{QCR}(t, z) = \frac{\pi}{c^2} j^2(t_{ret}), \tag{24}$$

where $t_{ret} = t - z/c$ — delay time and $\mathbf{j}(t)$ — $D = 2 + 1$ current density in graphene. As expected, the quasiclassical radiation is relatively weak (Fig. 5).

The interaction of the EHP with the photon field generates quantum radiation (QR). The corresponding KE system for the EHP and the photon subsystem in graphene was obtained in Ref. [16] on a dynamic basis by the analogy with the kinetic theory of electron-positron-photon plasma in a strong external field [17,18].

We will consider below only the annihilation channel in the integral of collisions of photonic KE, neglecting the reverse effect of the birth of electron-hole pairs as a result of photon absorption [16],

$$\dot{F}(\mathbf{K}, t) = 2\int \frac{d^2p}{(2\pi\hbar)^2} \int_{t_0}^{t} dt' K_\gamma(\mathbf{p}, \mathbf{p} + \hbar\mathbf{k}; t, t') \times \Big\{ f(\mathbf{p}, t') f(\mathbf{p} + \hbar\mathbf{k}, t') + \big[f(\mathbf{p}, t') + f(\mathbf{p} + \hbar\mathbf{k}, t') - 1 \big] \times F(\mathbf{K}, t') \Big\}, \tag{25}$$

where the vector $\mathbf{k}$ belongs to the graphene plane, $\mathbf{K} = (\mathbf{k}, k^{(3)})$ — a three-dimensional wave vector with a component $k^{(3)}$, orthogonal to the graphene plane; $K = |\mathbf{K}|$. The kernel of the photon collision integral here is equal to

$$K_\gamma(\mathbf{p}, \mathbf{p}'; t, t') = \frac{(ev_F)^2}{2\hbar cKd} \Gamma_{uv}^{\alpha}(\mathbf{p}, \mathbf{p}'; t) \Gamma_{uv}^{\alpha *}(\mathbf{p}, \mathbf{p}'; t') \times \cos\Theta^{(+)}(\mathbf{p}, \mathbf{p}'; t, t'). \tag{26}$$

Phase $\Theta^{(+)}(\mathbf{p}, \mathbf{p}'; t, t')$ corresponds to a single-photon annihilation diagram

$$\Theta^{(+)}(\mathbf{p}, \mathbf{p}'; t, t') = \frac{1}{\hbar} \int_{t'}^{t} d\tau \, [\varepsilon(\mathbf{p}, \tau) + \varepsilon(\mathbf{p}', t) - c\hbar K]. \tag{27}$$

In case of a sufficiently slow process it is possible to neglect the delay in the product of vertex functions in (26) and use the relation [16]:

$$\sum_{\alpha} |\Gamma_{uv}^{\alpha}(\mathbf{p}, \mathbf{p}'; t)| = 1. \tag{28}$$

The time integral in (25) then leads to the law of conservation of energy in the elementary act of annihilation (a similar situation occurs in case of derivation of the kinetic Boltzmann equation [19]). The process is inelastic in the considered case and the law of conservation of energy does not hold in the collision integral (25). The kernel of the collision integral in KE (25) in the approximation (28) will be equal to

$$K_\gamma(\mathbf{p}, \mathbf{p}'; t, t') = \frac{(ev_F)^2}{2\hbar cKd} \cos\Theta^{(+)}(\mathbf{p}, \mathbf{p}'; t, t'). \tag{29}$$

The appearance of the thickness d of the graphene sample in the resulting collision integral here is caused by the use of the same normalization volume $V = Sd$ in the expansion of the vector potential over plane waves

$$\hat{A}^{(\pm)\alpha}(\mathbf{x}, t) = \sqrt{\frac{\hbar c}{V}} \sum_{\mathbf{K}} \frac{1}{2\sqrt{2K}} \epsilon_i^{\alpha}(\pm\mathbf{K}) \hat{A}^{(\pm)}(\pm\mathbf{K}, t) e^{-i\mathbf{kx}}. \tag{30}$$

Here α, $i = 1, 2$, and ϵ_i^{α} — a polarization tetrad transverse to the vector $\mathbf{K}$. The final KE system in the electron-hole and photon sectors is recorded in the thermodynamic limit $V \to \infty$ with a fixed sample thickness. As a result,

the photonic KE in the annihilation channel (25) has the following form

$$\dot{F}(\mathbf{K},t)=\Lambda(K)\int\frac{d^2p}{(2\pi\hbar)^2}\int_{t_0}^{t}dt'\cos\Theta^{(+)}(\mathbf{p},\mathbf{p}+\hbar\mathbf{k};t,t')$$
$$\times\left\{f(\mathbf{p},t')f(\mathbf{p}+\hbar\mathbf{k},t')+[f(\mathbf{p},t')+f(\mathbf{p}+\hbar\mathbf{k},t')-1]\right.$$
$$\left.\times F(\mathbf{K},t')\right\}=C(\mathbf{K},t), \tag{31}$$

where $C(\mathbf{K},t)$ — collision integral and

$$\Lambda(K)=\frac{(ev_F)^2}{\hbar cKd}. \tag{32}$$

Given the similarity of the mathematical structures of the integro-differential equations (2) and (31), it is not difficult to write the photonic equation (31) in the form of a system of integro-differential equations similar to (3):

$$\dot{F}(\mathbf{K},t)=\Lambda(K)\int\frac{d^2p}{(2\pi\hbar)^2}U(\mathbf{p},\mathbf{K},t),$$
$$\dot{U}(\mathbf{p},\mathbf{K},t)=[f(\mathbf{p},t')+f(\mathbf{p}+\hbar\mathbf{k},t')-1]F(\mathbf{K},t')+f(\mathbf{p},t')$$
$$\times f(\mathbf{p}+\hbar\mathbf{k},t')-\frac{1}{\hbar}\left[\varepsilon(\mathbf{p},\tau)+\varepsilon(\mathbf{p}',t)-c\hbar K\right]V(\mathbf{p},\mathbf{K},t),$$
$$\dot{V}(\mathbf{p},\mathbf{K},t)=\frac{1}{\hbar}\left[\varepsilon(\mathbf{p},\tau)+\varepsilon(\mathbf{p}',t)-c\hbar K\right]U(\mathbf{p},\mathbf{K},t). \tag{33}$$

This system is convenient for numerical study of the quantum radiation problem. The distribution function of carriers $f(\mathbf{p},t)$ is considered here as a solution of KE (2)(or the system (3)).

We select the parameters of the external field in the model (19) as $E_0=250\,\text{kV/cm}$ and $\tau=246\,\text{fs}$, which they are close to the parameters of the work field [20] for comparing the obtained results of the characteristics of quasiclassical radiation and quantum radiation based on the kinetic theory with the experimental results [20].

Spectral power densities of quasiclassical radiation ($Q_{QCR}=\dot{\mathscr{E}}_{QCR}$, (24)) and quantum radiation

$$Q_{QR}(\nu)=\frac{4\pi\hbar}{c^3}\nu^3\dot{F}(\nu) \tag{34}$$

are shown in Fig. 5. The quasiclassical radiation field has a narrow spectrum with a sharp boundary in the high-frequency region, whereas the quantum radiation spectrum is much broader and reaches the ultraviolet spectral region. Only the quantum radiation reaches the visible light range in the considered situation. This result is close to the experimental observation [20].

The radiation is emitted in both directions from the graphene surface and orthogonally to it.

The situation shown in Fig. 5 is not universal and can be very different depending on the parameters of the external field (Fig. 6). The presented theory makes it possible to make the necessary predictions in these cases.

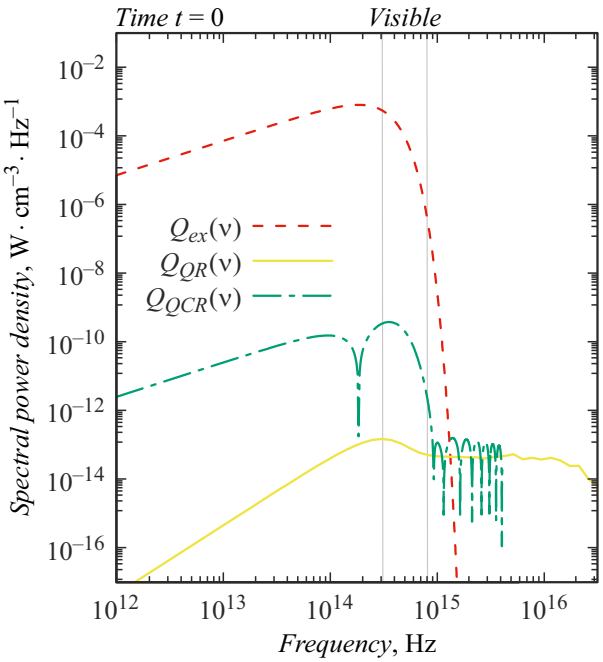

Figure 6. Spectral power densities of radiation with $E_0=100\,\text{kV/cm}$, $\tau=1.23\cdot10^{-15}$ c.

It should be noted that radiation (quasiclassical and quantum) leads to irreparable losses of energy introduced by the external field. These losses are not taken into account in this paper.

Conclusion

The work briefly summarizes the results of the development of a consistent kinetic theory of radiation processes in graphene, including a nonperturbative description of the generation of EHP, the mechanism of the back-reaction during the generation of internal currents and a quasiclassical plasma field, as well as a quantum field excited by direct interaction with carriers in the annihilation channel. An essential element of such a description is the use of QED methods [10], which allow combining $D=2+1$dynamics of graphene with $D=3+1$ dynamics of quasiclassical radiation and quantum radiation. The achieved level of radiation description in graphene suggests a qualitative agreement with the experiments [20]. It is possible to expect that further development of the kinetic theory (for example, taking into account the process of the back-reaction of the photonic subsystem in the annihilation channel) will allow reaching a quantitative level of description of radiation processes in graphene. In particular, it would be interesting to compare the mechanisms of depletion of the external field in graphene and the simplest semiconductor model, where a nonperturbative kinetic description of electron-hole excitations is also possible [21,22].

Appendix

Plasma currents in graphene generate quasiclassical radiation into regions of space external to the plane of S graphene. The characteristics of this radiation can be found using a model of spatially homogeneous currents in an infinite conducting plane [23].

The expression for the delayed vector potential $A^k_{ret}(r,t)$ of the radiation field in point z located at a distance z from the graphene plane is the initial expression [24].

$$A^k_{ret}(r,t) = \frac{1}{c}\int\limits_{S_d} \frac{d^3x}{\rho}\, \tilde{j}^{(k)}(t-\rho/c) = \frac{1}{c}\int\limits_{S} j^{(k)}(t-\rho/c). \tag{A1}$$

where $\rho = \sqrt{r^2+z^2}$, $r^2 = x^2+y^2$, and d — thickness of the graphene sample. The last equality (A1) is written taking into account the matching rule (17). Since now $rdr = \rho d\rho$, we derive the following from (A1).

$$A^k_{ret}(r,t) = \frac{2\pi}{c}\int\limits_{z}^{\infty} j^{(k)}(t-\rho/c). \tag{A2}$$

The strength of the electric and magnetic fields in the observation point z can be derived from (A2) by introducing a single vector $\mathbf{e}^{(3)}$ in the direction perpendicular to the plane S

$$\mathbf{E}_{rad}(t_0) = -\frac{2\pi}{c}\,\mathbf{j}(t_0), \tag{A3}$$

$$\mathbf{B}_{rad}(t_0) = -\frac{2\pi}{c}\,[\mathbf{e}^{(3)}, \mathbf{j}(t_0)], \tag{A4}$$

where $t_0 = t - z/c$ — delay time. The expression for the Poynting vector follows from here

$$\mathbf{S}_{rad}(t_0) = c\mathscr{E}_{rad}(t_0)\mathbf{e}^{(3)}, \tag{A5}$$

where the energy density of the quasiclassical radiation is

$$\mathscr{E}_{rad}(t_0) = \frac{\pi}{c^2}\, j^2(t_0). \tag{A6}$$

Let us now consider the quantum radiation in the outer regions of space („up“ and „down“) relative to the graphene plane S.

Since the point at issue is the extension of the kinetic description of photonic radiation to external regions, it is advisable to use the mathematical physics method of the Green's function, considering the graphene plane as an active radiation zone described by the photonic KE (31).

The photonic KE (31) with the collision integral $C(\mathbf{K},t)$:

$$\dot{F}(\mathbf{K},t) = C(\mathbf{K},t), \tag{A7}$$

which is valid on graphene planes S, is extended to both areas of free space ($\mathbf{X} = \mathbf{x},\ x^{(3)} = z$),

$$\hat{L}(\mathbf{X},t)F(\mathbf{X},\mathbf{K},t) \equiv \left(\frac{\partial}{\partial t} + c\mathbf{e}\,\frac{\partial}{\partial \mathbf{X}}\right) F(\mathbf{X},\mathbf{K},t)$$
$$= Q(\mathbf{X},\mathbf{K},t) \equiv C(\mathbf{K},t)\delta(z/d), \tag{A8}$$

where $\mathbf{e} = \mathbf{K}/K$. Now let's define the Green's function of the equation (A8)

$$\hat{L}(\mathbf{X},t)G(\mathbf{X}-\mathbf{X}',t-t') = \delta(\mathbf{X}-\mathbf{X}')\delta(t-t'). \tag{A9}$$

The Fourier image of the Green's function of the equation ($\varepsilon > 0$) can be obtained from here:

$$G(\mathbf{K},\omega) = \frac{-i}{\omega + i\varepsilon - cK}. \tag{A10}$$

Now it is possible to write the solution of KE (A8) via the source function $Q(\mathbf{X},\mathbf{K},t)$,

$$F(\mathbf{X},\mathbf{K},t) = \int d^3X dt' G(\mathbf{X}-\mathbf{X}',t-t')Q(\mathbf{X}',\mathbf{K},t'). \tag{A11}$$

We derive from here the following using the definition $Q(\mathbf{X},\mathbf{K},t)$ (A8) and formula (A10)

$$F(\mathbf{X},\mathbf{K},t) = \frac{d}{2c}\,C(\mathbf{K},t_{ret}), \tag{A12}$$

where $t_{ret} = t - z/c$ — delay time.

Conflict of interest

The authors declare that they have no conflict of interest.

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