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Comparative analysis of classical and fractional equations of motion

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The properties of dissipative systems, which are described by classical and fractional equations of motion, are analyzed. The Lagrange functions of these systems are presented and compared. Examples of infinitive and finite motion are considered. Based on the results obtained, a clarifying interpretation of fractional integro-differentiation in problems of mechanics is given.

Keywords: Lagrange equation, Lagrange function, equation of motion, energy dissipation, infinite and finite motions, fractional integro-differentiation.

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The total energy in any real dynamic system always decreases by transforming into other types of energy (specifically, heat). This is caused by resistance forces that are inevitably present in a system. An open system essentially ceases to be dissipative if the energy lost in it is compensated fully by the energy supplied externally. In mechanics, dissipation is normally taken into account by introducing the Rayleigh dissipative function, which characterizes the rate of reduction of the system energy, into the Lagrange equation [1]. Two new features arise in theory when this approach is applied [2]: (1) time reversal symmetry is violated; (2) the principle of least action ceases to be valid. The first property is considered to be fundamental and related to the irreversibility of real dynamic processes. The second property depends on the way dissipation is treated; generally speaking, it may be derived from the theoretical framework. One approach providing an opportunity to do just that involves the explicit introduction of time irreversibility into the Lagrange equation.

In the present study, the Bateman–Caldirola–Kanai (BCK) method [2,3] and the method relying on fractional integro-differentiation [4,5] are compared in the context of dissipative mechanical systems. Both methods involve the application of the time-dependent Lagrange function. One-dimensional motion of a material particle is considered. The starting point is the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0, \tag{1}$$

where $L = L(t, x, v)$ is the Lagrange function that depends on time t and coordinate x and velocity v of the material particle.

In the BCK method, an exponential factor is added to the Lagrange function of a conservative system [2,3]:

$$L = \left[\frac{mv^2}{2} - U(x, t)\right] \exp(t/\tau), \tag{2}$$

where $U(x, t)$ is the potential function, m is mass, and $\tau > 0$ is a certain characteristic time. Inserting (2) into (1), we obtain the equation of motion

$$m\frac{dv}{dt} + \frac{1}{\tau}mv = -\frac{\partial U(x, t)}{\partial x}. \tag{3}$$

The second term on the left-hand side of (3) specifies the force of dynamic friction. Note that the BCK method is used most often in the theory of quantum dissipative systems [6,7].

Let us assume that a force acting on a mechanical system may be presented in the form

$$F(x, t) = -\frac{\partial}{\partial t}\int_0^t \left(\frac{\partial U(x, t')}{\partial x}\right) G(t - t')dt', \tag{4}$$

where $G(t)$ is the dynamic memory function. Function $G(t)$ characterizes the change in momentum in response to a short-term force action. Formula (4) is tantamount to considering a certain nonlocal potential function $\tilde{U}(x, t)$ that takes a lag in interaction into account and is expressed in terms of the convolution integral:

$$\tilde{U}(x, t) = U_0 + \frac{\partial}{\partial t}\int_0^t U(x, t')G(t - t')dt', \tag{5}$$

where U_0 is, in general, time-dependent. Taking (5) into account, we write the Lagrange function

$$L = \frac{mv^2}{2} - \tilde{U}(x, t). \tag{6}$$

Inserting (5) and (6) into (1), we obtain the equation of motion

$$m\frac{dv}{dt} = -\frac{\partial}{\partial t}\int_0^t \left(\frac{\partial U(x, t')}{\partial x}\right) G(t - t')dt'. \tag{7}$$

Numerous real systems feature a power-law dynamic memory [5] that is specified by a function of the form

$$G(t) = \frac{1}{\Gamma(\alpha)}\left(\frac{\tau}{t}\right)^{1-\alpha}, \qquad 0 < 0 \leqslant 1. \tag{8}$$

The following equation of motion is derived from (7) and (8):

$$m\frac{dv}{dt} = -\frac{\tau^{1-\alpha}}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int\limits_0^t \left(\frac{\partial U(x,t')}{\partial x}\right)\frac{dt'}{(t-t')^{1-\alpha}}. \tag{9}$$

If $|\partial U(x,t)/\partial x| \leqslant t^{-\varepsilon}C(x)$ for a certain $\varepsilon < \alpha$ and time-independent function $C(x)$, the following limit relation holds true:

$$\lim_{t\to 0}\int\limits_0^t \left(\frac{\partial U(x,t')}{\partial x}\right)\frac{dt'}{(t-t')^{1-\alpha}} = 0.$$

This relation allows one to rewrite Eq. (9) in a more compact form:

$$\frac{m}{\tau^{1-\alpha}}\frac{d^\alpha v}{dt^\alpha} = -\frac{\partial U(x,t)}{\partial x}, \tag{10}$$

where [4,5]:

$$\frac{d^\beta v}{dt^\beta} = \frac{1}{\Gamma(n-\beta)}\int\limits_0^t \frac{v^{(n)}(t')dt'}{(t-t')^{\beta-n+1}}$$

$$(n-1 < \beta \leqslant n, n \in \mathbb{N}).$$

Contrary to the first impression, the fractional derivative in (10) does not specify a „fractional acceleration." This is seen easily from the Lagrange function (6) and equation of motion (9). Equation of motion (10) is presented in a convenient mathematical form, but may always be transformed to the form with a common acceleration and an effective force. If one relies on the principle of least action and the Lagrange equation (1), Eq. (10) currently appears to be better substantiated and preferable to the equation with a fractional force of friction (see, e.g., review [8]) for which the Lagrangian and/or the Rayleigh dissipative function have not been constructed yet. Compared to function (2), function (6) has a form that is more conventional for classical mechanics. If one uses (6), it is trivial to write Hamilton–Jacobi and Schrödinger equations.

Thus, the task is to analyze equations of motion (3) and (10). Dissipation is introduced into Eq. (3) as an additional term that specifies the force of dynamic friction. In Eq. (10), dissipation is characterized with the use of dynamic memory function (8) and a fractional derivative. In mathematical terms, an increase in dissipation corresponds to an enhancement of contribution of the lowest derivative in Eq. (3) and a reduction of the order of Eq. (10) via parameters τ and α, respectively.

In what follows, Eqs. (3) and (10) are used to analyze examples of infinite and finite motion: the motion of charged particles under the influence of constant and variable electric fields. The motion of an electron in a constant electric field is characterized by equations

$$m\frac{dv}{dt} + \frac{1}{\tau}mv = qE, \tag{11}$$

$$\frac{m}{\tau^{1-\alpha}}\frac{d^\alpha v}{dt^\alpha} = qE, \tag{12}$$

where q is the electron charge and $E = \mathrm{const}$ is the electric field intensity. It is easy to see that these two equations are exactly matching only at $t \to \infty$, $dv/dt = 0$, and $\alpha = 0$. The following relation known from the Drude model of electrical conduction of metals is obtained in this case from (11), (12):

$$v = \frac{q\tau}{m}E = \mu E,$$

where μ is the electron mobility. A similar conclusion is derived in the problem of a free-falling body [5,9]. The presence of a restoring force is typical of finite motion. Specifically, the displacements of oscillator atoms of a solid in variable electric field $E_0\exp(-i\omega t)$ are characterized by equation

$$\frac{d^2x}{dt^2} + \frac{1}{\tau}\frac{dx}{dt} + \omega_0^2 x = \frac{qE_0}{m}\exp(-i\omega t), \tag{13}$$

where E_0 and ω are the magnitude and the frequency of an external variable electric field and ω_0 is the natural frequency of oscillators. The solution of Eq. (13) is sought in the form $x = x_0\exp(-i\omega t)$. The polarization is $P = \varepsilon_0(\varepsilon - 1)E_0\exp(-i\omega t) = qNx_0\exp(-i\omega t)$. It follows that permittivity may be written as

$$\varepsilon = 1 + \frac{qNx_0}{\varepsilon_0 E_0}. \tag{14}$$

With the solution of Eq. (13) taken into account, we obtain the following from (14):

$$\varepsilon = 1 - \frac{(\omega_p/\omega_0)^2}{(\omega/\omega_0)^2 + i/(\omega_0\tau) - 1}, \tag{15}$$

where $\omega_p = \sqrt{q^2N/(\varepsilon_0 m)}$ is the plasma frequency. At the same time, we have a fractional equation of motion

$$\frac{d^{1+\alpha}x}{dt^{1+\alpha}} + \omega_0^{1+\alpha}x = \frac{qE_0}{m\omega_0^{1-\alpha}}\exp(-i\omega t). \tag{16}$$

The general solution of Eq. (16) is written as [4, p. 17]

$$x(t) = AE_{\alpha+1,1}\big(-(\omega_0 t)^{1+\alpha}\big) + BtE_{\alpha+1,2}\big(-(\omega_0 t)^{1+\alpha}\big)$$
$$+\frac{qE_0}{m\omega_0^{1-\alpha}}\int\limits_0^t E_{\alpha+1,2}\big(-(\omega_0 s)^{1+\alpha}\big)\exp\big(i\omega(s-t)\big)s^\alpha ds, \tag{17}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [4,5,10–13]. It follows from (17) that

$$x_0(\omega) = \lim_{t\to\infty} x(t)\exp(i\omega t) = \frac{qE_0}{m\omega_0^{1-\alpha}}\frac{1}{(-i\omega)^{1+\alpha} + \omega_0^{1+\alpha}}.$$

Inserting this expression into (14), we find

$$\varepsilon = 1 - \frac{(\omega_p/\omega_0)^2}{(-i\omega/\omega_0)^{1+\alpha} + 1}. \quad (18)$$

Formula (18) is similar in appearance to the known Cole−Cole equation [5,14]. The only difference is in the exponent of power, which is less than one in the Cole−Cole formula. The results of numerical calculations of real and imaginary parts of dielectric functions (15) and (18) reveal that the classical and fractional models are equally applicable to spectra with a narrow and symmetric absorption peak. It has been demonstrated in our studies [10–13] that the order of the fractional derivative in the oscillation equation is related to the Q factor. The parameters in Eqs. (13) and (16) are related in the same way: $\alpha \approx 1 - 2/(\pi\omega_0\tau)$.

An interesting conclusion was made in [9,15]: a fractional integro-differential operator emerges naturally in the examination of an open mechanical system interacting with its surroundings. This conclusions is consistent, to some extent, with our analysis, where potential function (5) is used to reproduce the open and dissipative character of a mechanical system, which is manifested in interaction lag.

Let us summarize briefly the key findings. Equations of motion (3) and (10), which are based on the Lagrange equation (1) and include the dissipation of energy, were examined. These equations have different solutions; a comparison between them is meaningful only in the case of steady motion (i.e., at $t \to \infty$). It has been noted in [2] that if the right-hand side of an equation of motion does not depend explicitly on time, the problem may always be reduced to Lagrange or Hamilton equations. It was demonstrated above that an equation of motion with a fractional time derivative may also be reduced to these equations.

Conflict of interest

The authors declare that they have no conflict of interest.

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