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On the area of extremely short electromagnetic pulses in medium for problems of a finite integration region

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Based on the one-dimensional Maxwell equations of electrodynamics of continuous medium, the balance ratios of electric and magnetic areas for problems solved in a finite region in space and time are obtained.

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In the study of extremely short light pulses, it is no longer possible to use the approximation of a slowly varying envelope of the electric field intensity variable $\mathscr{E}(t)$. With the loss of the concept of the pulse envelope, the pulse area tool $\theta = \frac{d_{12}}{\hbar}\int\limits_{-\infty}^{\infty}\mathscr{E}(t)dt$ does not work, here d_{12} — dipole transition point a two-level atom, $\hbar$ — Planck's constant. But the concept „electric area" $S_E = \int\limits_{-\infty}^{\infty} E dt$ (in a dimensionless form $s_e = \frac{2d_{\{12\}}}{\hbar}S_E$) is meaningful. It was found in the articles [1–4] that the quantity $S_E(z)$ does not change with the change in the spatial coordinate z along which the electromagnetic radiation propagates. The articles [3,4] provide conditions under which integration with infinite limits in calculation of $S_E(z)$ is replaced by finite limits $S_E = \int\limits_{t_{\{1\}}}^{t_2} E dt$. Here is an excerpt from the article [3]: „the radiation wave packet at the initial time is located in vacuum, there is no field in the medium, and after a sufficiently long time interval after the packet passes through, the field disappears at any fixed point in space".

Based on the one-dimensional Maxwell equations of electrodynamics of continuous medium, we will obtain the balance ratios of electric and magnetic areas for problems solved in a finite region both in space and time. For the one-dimensional geometry of the problem of electromagnetic radiation propagation in a dielectric, we use the Maxwell equations

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial H}{\partial t} = 0, \tag{1}$$

$$\frac{\partial H}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{4\pi\dot{P}}{c}. \tag{2}$$

Polarized radiation propagates along the z axis, the electric field intensity E is directed along the x axis, and the magnetic field intensity H— along the y axis, t — time. The electrodynamic combined equations (1), (2) have a solution in the form of electromagnetic waves. They propagate in both positive and negative directions.

The problem is solved in a rectangular region D_p of two independent variables z and t, $z_L \le z \le z_R$, $t_B \le t \le t_E$. Everything that follows will be valid for any rectangular subregion D of a region D_P limited by z_1, z_2 such that $z_L \le z_1 < z_2 \le z_R$, and t_1, t_2 — $t_B \le t_1 < t_2 \le t_E$. The regions D are shown in Fig. 1 and 2 (a).

Let's introduce the designations

$$S_{E,z} = \int\limits_{t_1}^{t_2} E(z,t)\,dt, \qquad S_{H,z} = \int\limits_{z_1}^{z_2} \frac{H(z,t)}{c}\,dz, \tag{3}$$

$$S_{H,z} = \int\limits_{t_1}^{t_2} H(z,t)\,dt \quad S_{E,z} = \int\limits_{z_1}^{z_2} \frac{E(z,t)}{c}\,dt,$$

$$S_{P,z} = \int\limits_{z_1}^{z_2} \frac{4\pi P(z,t)}{c}\,dz. \tag{4}$$

In these formulas, the subscripts have the following meaning: the first index is the integrand, the second index is one of the independent variables, which remains constant during integration. Integration is carried out over the remaining independent variable.

Then the following two statements are true for any subregion D:

$$(S_{E,z_2} - S_{E,z_1}) + (S_{H,t_2} - S_{H,t_1}) = 0 \tag{5}$$

$$(S_{H,z_2} - S_{H,z_1}) + (S_{E,t_2} - S_{E,t_1} + S_{P,t_2} - S_{P,t_1}) = 0. \tag{6}$$

Let us prove the first of them. Let us take the double integral of the left-hand side (1) over the region and apply Green's formula. We obtain

$$\iint\limits_{D_K}\left(\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial H}{\partial t}\right)dz\,dt = \oint\limits_K \left(-\frac{1}{c}H dz + E dt\right). \tag{7}$$

The integrand on the left side of this formula is equal to zero, therefore, the double integral is equal to zero. So the right curvilinear integral is also equal to zero. Let us write the curvilinear integral on the right side of (7) taking into account the direction of integration

$$\oint_K \left(-\frac{1}{c}Hdz + Edt\right) = \int_{t_1}^{t_2} E(z_2, t)dt - \int_{t_1}^{t_2} E(z_1, t)dt$$

$$+ \int_{z_1}^{z_2} \frac{H(z, t_2)}{c}dz - \int_{z_1}^{z_2} \frac{H(z, t_1)}{c}dz = 0.$$

Taking into account (3), we obtain (5).

To prove the validity of (6), we transform (2) to the following form:

$$\frac{\partial H}{\partial z} + \frac{1}{c}\frac{\partial(E + 4\pi P)}{\partial t} = 0.$$

The further proof is carried out in the same way as in the proof (5). The expression in brackets in (8) is the electric induction vector $D = E + 4\pi P$, which allows us to write (6) in short form

$$(S_{H,z_2} - S_{H,z_1}) + (S_{D,t_2} - S_{D,t_1}) = 0. \tag{9}$$

Here $S_{D,t} = \int_{z_1}^{z_2} \frac{D(z,t)}{c}dz = \int_{z_1}^{z_2} \frac{E(z,t)+4\pi P(z,t)}{c}dz$.

Researchers often use dimensionless variables. For a two-level medium, we introduce dimensionless variables [5,6] $\tau = t/t_0$, $\xi = z/z_0$, $e = E/E_0$, $h = H/H_0$, $p = P/P_0$, here $t_0 = \omega_0^{-1}$, $z_0 = c/\omega_0$, $E_0 = H_0 = P_0 = \hbar\omega_0/2d$. Maxwell's equations are written as

$$\frac{\partial e}{\partial \xi} + \frac{\partial h}{\partial \tau} = 0, \qquad \frac{\partial h}{\partial \xi} + \frac{\partial(e + 4\pi p)}{\partial \tau} = 0.$$

Having performed the above actions, we get the following relations instead of (5), (6), (9):

$$(s_{e,\xi_2} - s_{e,\xi_1}) + (s_{h,\tau_2} - s_{h,\tau_1}) = 0, \tag{10}$$

$$(s_{h,\xi_2} - s_{h,\xi_1}) + (s_{e,\tau_2} - s_{e,\tau_1}) + (s_{4\pi p,\tau_2} - s_{4\pi p\tau_1}) = 0,$$

or

$$(s_{h,\xi_2} - s_{h,\xi_1}) + (s_{d,r_2} - s_{d,r_1}) = 0. \tag{11}$$

Here,

$$s_{e,\xi} = \int_{r_1}^{r_2} e(\xi, \tau)d\tau, \qquad s_{h,r} = \int_{\xi_1}^{\xi_2} h(\xi, \tau)d\xi,$$

$$s_{h,\xi} = \int_{\tau_1}^{\tau_2} h(\xi, \tau)d\tau, \quad s_{e,\tau} = \int_{\xi_1}^{\xi_2} e(\xi, \tau)d\xi,$$

$$s_{4\pi p,\tau} = \int_{\xi_1}^{\xi_2} 4\pi p(\xi, \tau)d\xi,$$

$$s_{d,\tau} = \int_{\xi_1}^{\xi_2} \big(e(\xi, \tau) + 4\pi p(\xi, \tau)\big)d\xi = \int_{\xi_1}^{\xi_2} d(\xi, \tau)d\xi.$$

Let us introduce the following designation: $r_{f,\xi} = (s_{f,\xi_2} - s_{f,\xi_1})$, this expression is the difference in areas with a larger value ξ_2 and with a smaller value ξ_1 of the region D. Then (10), (11) can be written in short form

$$r_{e,\xi} + r_{h\tau} = 0, \tag{12}$$

$$r_{h,\xi} + r_{d,\tau} = 0 \quad \text{or} \quad r_{h,\xi} + (r_{e,\tau} + r_{4\pi p,\tau}) = 0. \tag{13}$$

From these ratios it follows:

The first conclusion. In order for one of the differences in equalities (12), (13) to be equal to zero, it is necessary and sufficient that the second difference is also equal to zero.

For example, when modeling radiation transport in substance, the initial state of the medium is assumed to be unperturbed, and τ_2 is taken large enough for the medium to return to the initial (unperturbed) state, and all electromagnetic radiation to leave the medium. Then $r_{h,\tau} = 0$, $r_{d,\tau} = 0$, hence $r_{e,\xi} = 0$, $r_{h,\xi} = 0$. Here the law of electric and magnetic areas conservation is realized [1–4].

The second conclusion. If one of the differences is not equal to zero, then the other is also not equal to zero. In addition, these differences are equal in absolute value and opposite in sign.

Here equalities (12), (13) must be satisfied at any time $\tau_2 = \tau$. The equality must be satisfied even when there is an electromagnetic field in the medium or medium polarization.

In numerical simulation, we obtain an approximate solution. To control correctness of the solution obtained by the numerical method, the calculation of the balance of quantities that fall under the conservation laws, e.g. total energy conservation law. The total energy means the sum of the electromagnetic radiation energy and the energy accumulated by the substance when the medium is excited. The above-described allows us to additionally introduce a balance of two quantities: Δs_e, Δs_h (imbalances of the electric and magnetic fields areas).

$$\Delta s_e = (r_{e,\xi} + r_{h,\tau})_{nu}, \qquad \Delta s_h = (r_{h,\xi} + r_{d,\tau})_{nu}. \tag{14}$$

In relative view

$$\delta s_e = \frac{\Delta s_e}{\max(r_{e,\xi})}, \qquad \delta s_h = \frac{\Delta s_h}{\max(r_{h,\xi})}.$$

The subscript of imbalances is inherited from the first difference in parentheses (14).

Let us take a region D limited in space by variable $\xi_1 \le \xi \le \xi_2$, and in time by variable $\tau_1 \le \tau \le \tau_z$. Here τ_z is the current time of the problem, further we will denote

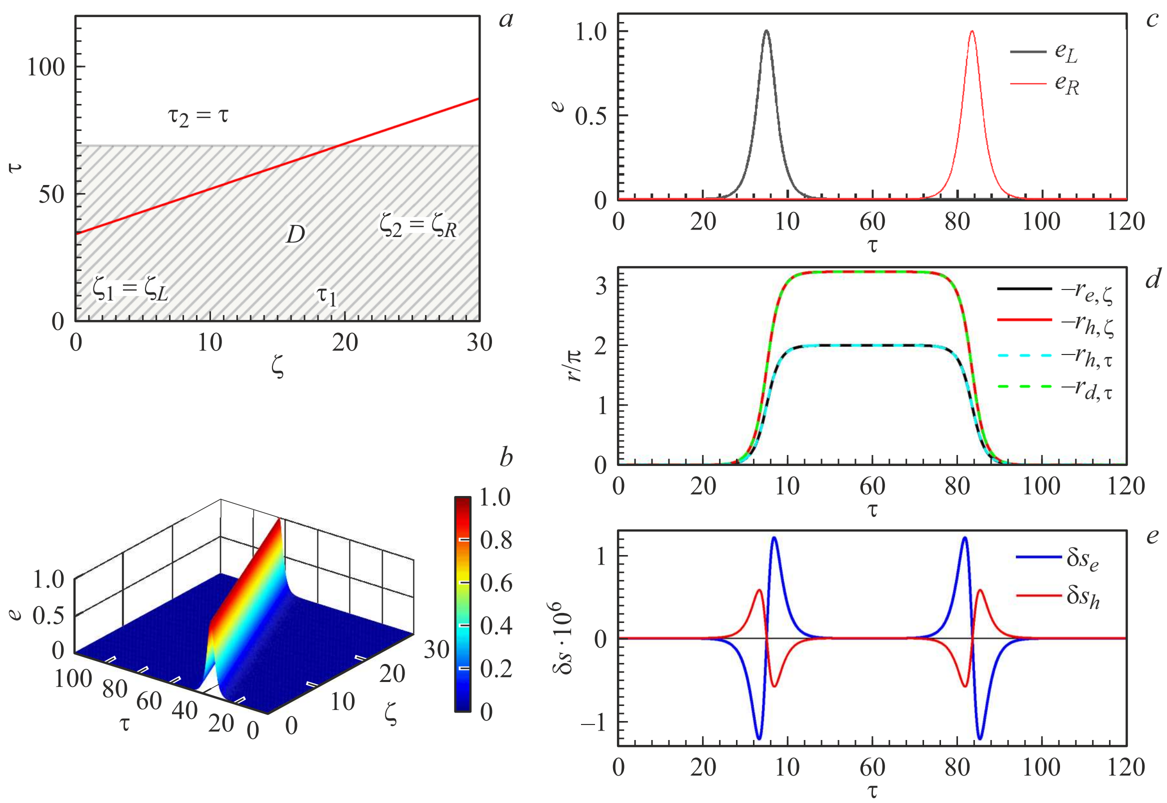

Figure 1. Problem I. Medium parameters: $N = 9 \times 10^{21}\,\text{cm}^{-3}$, $\omega_0 = 2.6909 \times 10^{15}$, $\lambda = 700\,\text{nm}$, $d = 5D$, $\alpha = 1$, $\gamma_1 = \gamma_2 = 0$; pulse duration $\tau_p = 2$. *a*) Region $D(\tau)$ on a plane ξ, τ, the inclined line is the pulse maximum trace. *b*) Dependence $e(\xi, \tau)$. *c*) Dependences $e(\tau)$ on the left and right boundaries. *d*) Functions $-r_{e\xi}$, r_{hr}, and $-r_{h,\xi}$, $r_{d,r}$. *e*) Imbalances $\delta s_e(\tau)$ and $\delta s_h(\tau)$.

it τ. Then „the area imbalances“ will be a function of the variable τ: $\delta s_e(\tau)\delta s_h(\tau)$.

When deriving the obtained results, only Maxwell electrodynamics equations were used as initial ones, the constitutive equations were not involved in the derivation. Therefore, we have the right to expect that the results obtained will be valid for any correct model of the medium. In the following two examples, we will use the Bloch equations for a two-level atom as a medium model: $ds_1/d\tau = -s_2 - \gamma_2 s_1$, $ds_2/d\tau = s_1 + es_3 - \gamma_2 s_2$, $ds_3/d\tau = -es_2 - \gamma_1 s_{e3}$, here $\gamma_i = (\omega_0 T_i)^{-1}$. Polarization $P(t, z) = N \cdot d \cdot s_1$.

The main results of solving Problem I and Problem II are shown in Figs. 1 and 2, respectively. For clarity, the parameters of the problems were selected to be extremely dense so that the parameter $\alpha = 4\pi N d^2/(\hbar\omega_0)$ was of the order of ≈ 1–6.

The first problem is about propagation of the unipolar pulse of self-induced transparency in a dense medium. This problem has an analytical solution [7]

$$e = \frac{2}{\tau_p}\,\mathrm{sech}\left(\frac{\tau - \tau_0 - \xi/\upsilon}{\tau_p}\right). \qquad (15)$$

In the problem, the electromagnetic radiation pulse is localized. Both scenarios can be observed, defined by both the first and the second conclusions.

As the second calculation, we took the problem from the articles [8,9], in which the interaction of unipolar counter-pulses of self-induced transparency (15) in dense medium was studied. As a result of the collision of two unipolar pulses of opposite polarity, a long-lived polariton cluster is formed, which has a double spatial structure. The diagram in Fig. 2, *b* shows the dependence $s_3(\xi, \tau)$.

Due to antisymmetry of the problem, the imbalances of the electric and magnetic areas turned out to be equal to zero identically, which indicates a good symmetrization of the numerical solution technique. Therefore, the region D was taken only by negative values ξ, the diagram of Fig. 2, *a*. On the diagram in Fig. 2, *c* only $e_L(\tau)$ is shown. On the right boundary of the region D due to the antisymmetry of the problem $e_R(\tau) = e(0, \tau) = 0$. This diagram shows the dependence of the magnetic field intensity at $\xi = 0$ at the symmetry center $h_R(\tau)$.

Diagrams (d) in both figures demonstrate how relations (12), (13) are satisfied. Let us transform (12), (13) to the form $-r_{e,\xi} = r_{h,\tau}$, $-r_{r,\xi} = r_{d,\tau}$, right and the left parts of these ratios are shown on the diagrams (d). Corresponding pairs of curves $-r_{e,\xi}(\tau)$, $r_{h,\tau}(\tau)$ and $-r_{h\xi}(\tau)$, $r_{d,\tau}(\tau)$ are not distinguishable on the diagrams. But there are still differences. This is shown by diagrams (e), where the dependences of the relative imbalances of areas $\delta s_e(\tau)$ and $\delta s_h(\tau)$ are shown. The cause of imbalances is the

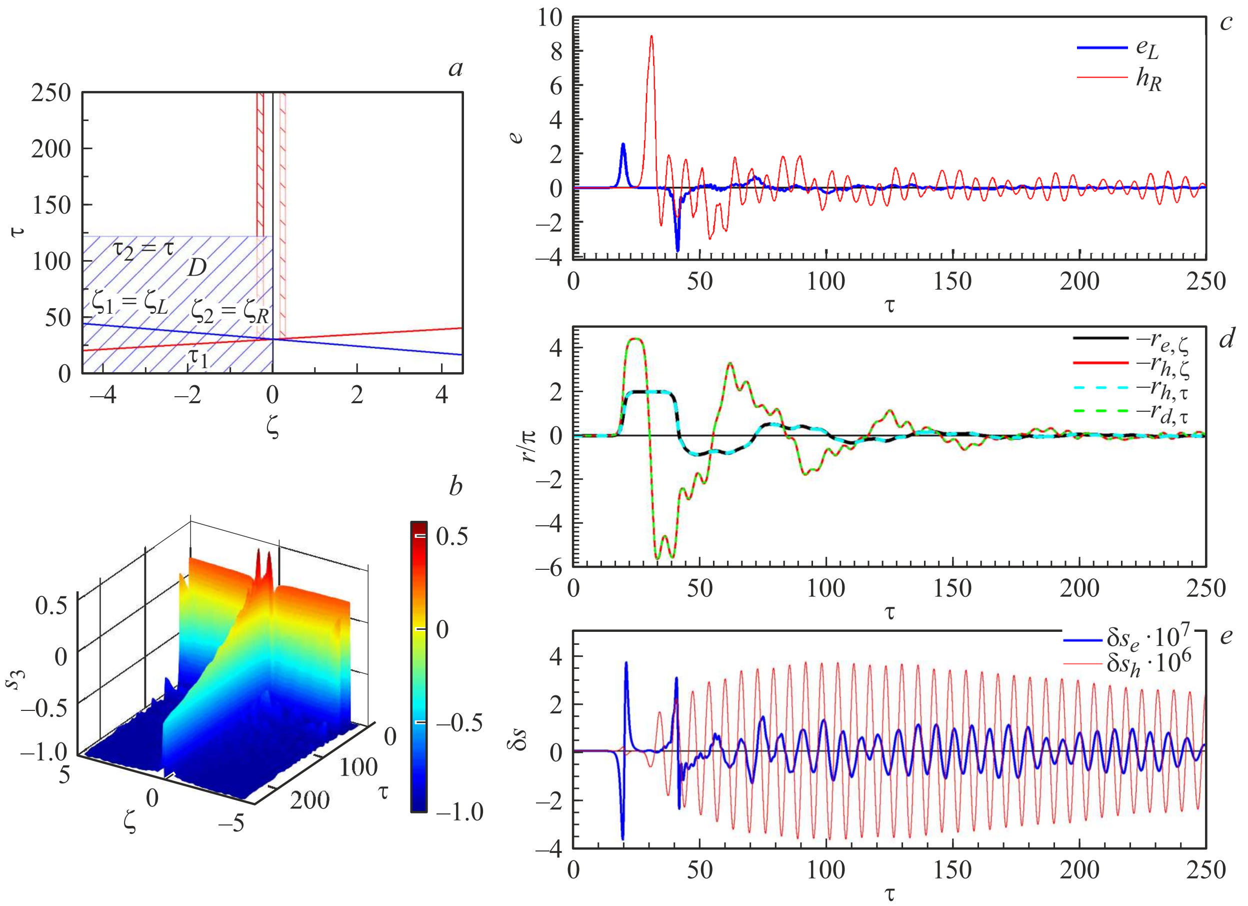

Figure 2. Problem II. Medium parameters: $N = 2.22 \times 10^{21}\,\text{cm}^{-3}$, $\omega_0 = 3.142\,\text{fs}^{-1}$ ($\lambda = 599.5\,\text{nm}$), $d = 25D$, $\alpha = 5.260$ ($\omega_c = 2\,\text{fs}^{-1}$), $T_1 = 1 \times 10^{-13}$, $T_2 = 0.5 \times 10^{-13}$. The pulse duration is $\tau_p = 0.78$. a) Region $D(\tau)$ on a plane ζ, τ, inclined lines are traces of counter-pulse maxima, polariton clusters are schematically depicted by narrow vertical regions. *b*) Dependence $s_3(\zeta, \tau)$. *c*) Dependence $e_L(\tau)$ on the left boundary and dependence $h_R(\tau)$ on the right boundary. *d*) Functions $r_{e,\xi}$, $r_{h,r}$ and $-r_{h,\xi}$, $r_{d,x}$. *e*) Imbalances $\delta s_e(\tau)$ and $\delta s_h(\tau)$.

Values of energy imbalances and maxima of area imbalances of electric and magnetic fields for problem II

Difference steps diagrams	δW	$\max \delta s_e$	$\max \delta s_h$
$10\Delta\tau_2$, $10\Delta z_2$	$1.26 \cdot 10^{-5}$	$5 \cdot 10^{-5}$	$0.37 \cdot 10^{-3}$
$\Delta\tau_2$, Δz_2	$1.27 \cdot 10^{-7}$	$4 \cdot 10^{-7}$	$0.4 \cdot 10^{-5}$
$0.1\Delta\tau_2$, $0.1\Delta z_2$	$1.27 \cdot 10^{-9}$	$5 \cdot 10^{-9}$	$0.4 \cdot 10^{-7}$

use of numerical methods for solving the Maxwell–Bloch equations. The second problem was calculated on three different grids for independent variables. The grid detail in the calculations changed by an order of magnitude. The table shows the energy imbalances δW of the maxima of the area imbalances $\max \delta s_e$ and $\max \delta s_h$, for problem II.

The data presented in the table confirm the second order of accuracy of the applied calculation method. Reducing the difference grid by an order of magnitude leads to a decrease in imbalances by two orders of magnitude.

Basically, the peak values of the imbalances arise when the radiation pulses pass through the boundaries of the medium. For the first problem, which has an analytical solution, even the dependence of the imbalance on time has an analytical expression — this is the derivative of the hyperbolic secant. In the second problem, until collision of counter-pulses, the development occurs as in the first problem. Starting from the moment when the pulses meet, the behavior of the imbalances $\delta s_e(\tau)$ and $\delta s_h(\tau)$ becomes oscillatory. At that, the oscillation amplitude increases and reaches $\max \delta s_h = 0.4 \cdot 10^{-5}$, but the imbalance remains quite acceptable. Moreover, we can always reduce it by splitting the difference grid.

Finally, the subregion D can consist of one, two, or four difference grid cells. In this case, relations (5), (6) can be a tool for building an integration-interpolation difference scheme for the numerical solution of problems [10].

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Conflict of interest

The authors declare that they have no conflict of interest.

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