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Laminar chaos in coupled time-delay systems

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The possibility of the existence of laminar chaos in coupled time-delayed feedback systems is investigated. The cases of unidirectional and mutual coupling of time-delay systems are considered. It is shown for the first time that laminar chaos can exist not only in a system with a variable delay time, but also in a system with a constant delay time, if it is coupled with a system in the regime of laminar chaos.

Keywords: Time-delay systems, Laminar chaos, Coupled oscillators, Synchronization

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The presence of delayed feedback is typical of systems of various nature [1,2]. This stimulates the interest in study of such systems in different branches of science [3,4]. The case with a constant delay time is considered in the majority of studies into time-delay systems. However, the examination of systems with a variable delay time is no less important, since the delay in real-world contexts may fluctuate for various reasons [5,6], thus making the system dynamics considerably more complicated [7,8]. The introduction of modulation of the delay time is useful in several practical applications. For example, the use of chaotic generators with a variable delay time in data-transmission systems allows one to make them more secure [9,10] than communication systems based on chaotic generators with delayed feedback and a constant delay time [11,12].

A new type of chaotic behavior (laminar chaos) has recently been identified in systems with a variable delay time [13]. This type of behavior is characterized by an interchange of different laminar phases. The dynamic variable remains almost constant within these phases, but changes chaotically in the transition from one laminar phase to another. The existence of laminar chaos has been demonstrated not only in numerical examples, but also in experimental systems: an optoelectronic generator with a delay varying in time [14], an electronic generator with a variable delay [15], and a radiotechnical generator with delayed feedback the delay time of which was modulated by an external harmonic signal [16].

Thus far, laminar chaos has been observed only in single self-oscillating time-delay systems under specific conditions of variation of the delay time [13–17]. In the present study, we demonstrate for the first time that laminar chaos may emerge not only in a system with a variable delay time, but also in a system with a constant delay time coupled with a system in the laminar chaos regime. Let us consider a time-delay system that, if not coupled with other systems, may be presented in the form of a ring consisting of three elements (nonlinear, inertial, and delay) and be characterized by the following first-order delay differential equation:

$$\varepsilon\dot{x}(t) = -x(t) + f\big(x(t-\tau)\big), \tag{1}$$

where ε is a parameter characterizing the inertial properties of the system, τ is the delay time, and f is a nonlinear function. The delay time in system (1) is constant. With a proper choice of the nonlinear function (e.g., quadratic or sine), system (1) demonstrates chaotic oscillations of dynamic variable $x(t)$ at $\tau \gg \varepsilon$ [1]. Since the $x(t)$ time series has no laminar sections with $x(t)$ remaining constant, these chaotic oscillations were called turbulent chaos in [13].

With a periodically varying delay time

$$\tau(t) = \tau_0 + \tau_m \sin(2\pi\nu t), \tag{2}$$

where τ_0 is the average value of the delay time, τ_m is the depth of modulation of the delay time, and ν is the modulation frequency, the system is characterized by equation

$$\varepsilon\dot{x}(t) = -x(t) + f\Big(x\big(t-\tau(t)\big)\Big) \tag{3}$$

and features a qualitatively different type of chaotic dynamics that was called laminar chaos [13,14] and is marked by the presence of horizontal plateaus in the $x(t)$ time series.

Let us now consider two coupled time-delay systems. One of them has a constant delay time and is in the turbulent chaos regime, while the other has a variable delay time and is in the laminar chaos regime. Time-delay systems may be coupled in various ways that differ both in the type of coupling (linear, diffusion, delayed) and in the position of the point within a ring time-delay system at which the signal from the other system is fed into the ring [18]. For example, the coupling signal may be fed into the system characterized by Eq. (1) between an inertial element (filter) and a delay line, between a delay line and a nonlinear element, or

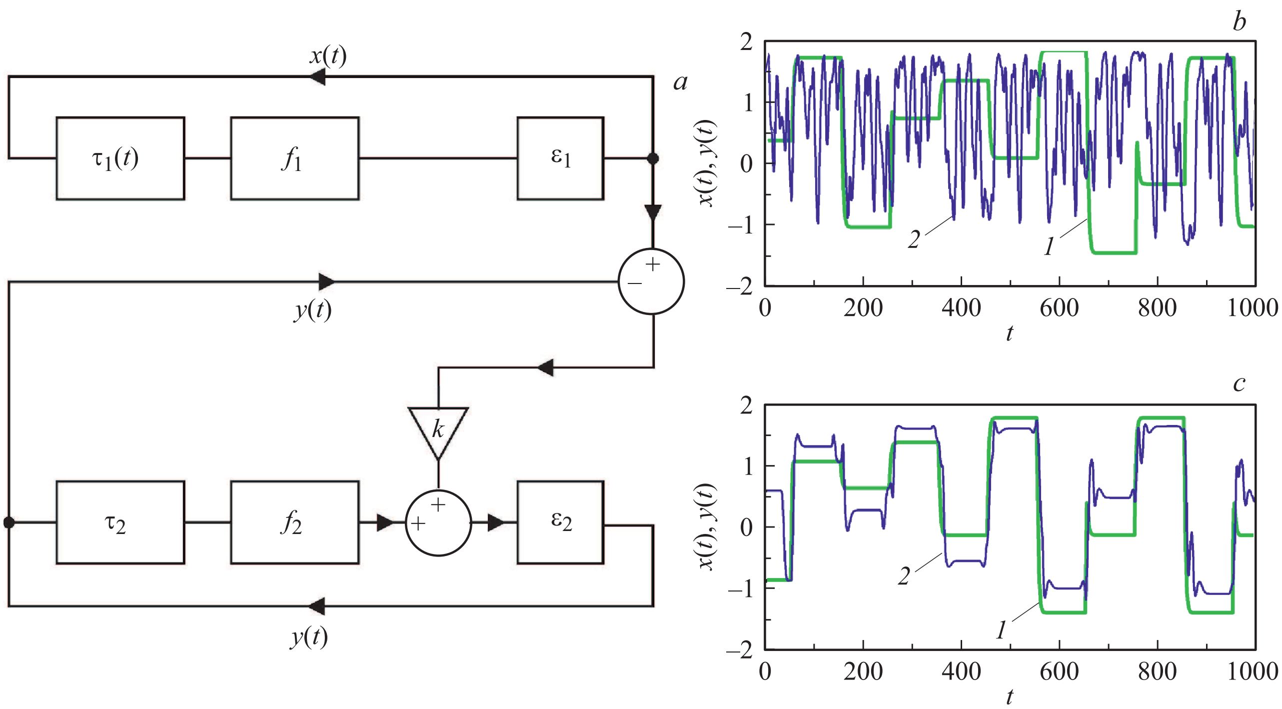

Figure 1. *a* — block diagram of two unidirectionally coupled time-delay systems. Elements $\tau_1(t)$ and τ_2, f_1 and f_2, and ε_1 and ε_2 implement delay, nonlinear transformation, and inertial transformation of oscillations, respectively. Element k defines the strength of unidirectional coupling. *b* and *c* — time series of $x(t)$ (*1*) and $y(t)$ (*2*) oscillations at $k = 0$ and 0.55, respectively.

between a nonlinear element and a filter. Each of these cases is characterized by its own equation [18].

Figure 1, *a* presents the block diagram of two unidirectionally coupled time-delay systems (1) and (3) for the case when the diffusion coupling signal is fed into the slave system between a nonlinear element and a filter. With this type of coupling, the master system is characterized by equation $\varepsilon_1\dot{x}(t) = -x(t) + f_1\big(x(t-\tau_1(t))\big)$, while the slave system is characterized by equation

$$\varepsilon_2\dot{y}(t) = -y(t) + f_2\big(y(t-\tau_2)\big) + k\big(x(t) - y(t)\big), \quad (4)$$

where k is the coupling coefficient.

The following parameters of master and slave systems were chosen: $\tau_1(t) = 1 + 0.2\sin(2\pi t)$, $\tau_2 = 1$, $\varepsilon_1 = \varepsilon_2 = 0.03$, $f_1 = \lambda_1 - x^2$, and $f_2 = \lambda_2 - y^2$, where $\lambda_1 = \lambda_2 = 1.82$ are the nonlinearity parameters; integration step $\Delta t = 0.01$. Figure 1, *b* presents the time series of $x(t)$ oscillations corresponding to the laminar chaos regime and $y(t)$ oscillations at $k = 0$ corresponding to the turbulent chaos regime. As the coupling strength increases, the turbulent chaos regime in the slave system starts collapsing. Sections of laminar chaos emerge in the $y(t)$ time series, and the duration of these sections increases with k. Figure 1, *c* presents the time series of $x(t)$ and $y(t)$ oscillations at $k = 0.55$. It is evident that the horizontal sections in the $y(t)$ time series are shorter than those in $x(t)$. In addition, sections of laminar chaos in the slave system at $k = 0.55$ are interspersed in the $y(t)$ time series with sections of turbulent chaos that are not shown in Fig. 1, *c*. Coefficient R of correlation between the master and slave systems is 0.82 at $k = 0.55$. The $x(t)$ and $y(t)$ oscillations become more and more similar as the coupling strength increases further. At $k > 1$, they are almost identical (i.e., the master and slave systems become synchronized completely). Correlation coefficient R is then close to unity.

Let us consider a more complicated case of mutual coupling between time-delay systems. Figure 2, *a* presents the block diagram of two mutually coupled time-delay systems (1) and (3) for the case when the diffusion coupling signal is fed into both systems between a filter and a delay line. With this type of coupling, the systems are characterized by equations

$$\varepsilon_1\dot{x}(t) = -x(t) + f_1\Big(x\big(t-\tau_1(t)\big) + k_1\big[y\big(t-\tau_1(t)\big) - x\big(t-\tau_1(t)\big)\big]\Big),$$

$$\varepsilon_2\dot{y}(t) = -y(t) + f_2\big(y(t-\tau_2) + k_2\big[x(t-\tau_2) - y(t-\tau_2)\big]\big), \quad (5)$$

where k_1 and k_2 are the coupling coefficients. The parameters of both systems were chosen to be the same as the ones used in the case of unidirectional coupling considered above. Therefore, the time series of $x(t)$ and $y(t)$ oscillations at $k_1 = k_2 = 0$ are identical to those presented in Fig. 1, *b*; i.e., the first system is in the laminar chaos regime, while the second system is in the turbulent chaos regime. Figure 2, *b* presents the time series of $x(t)$ and $y(t)$

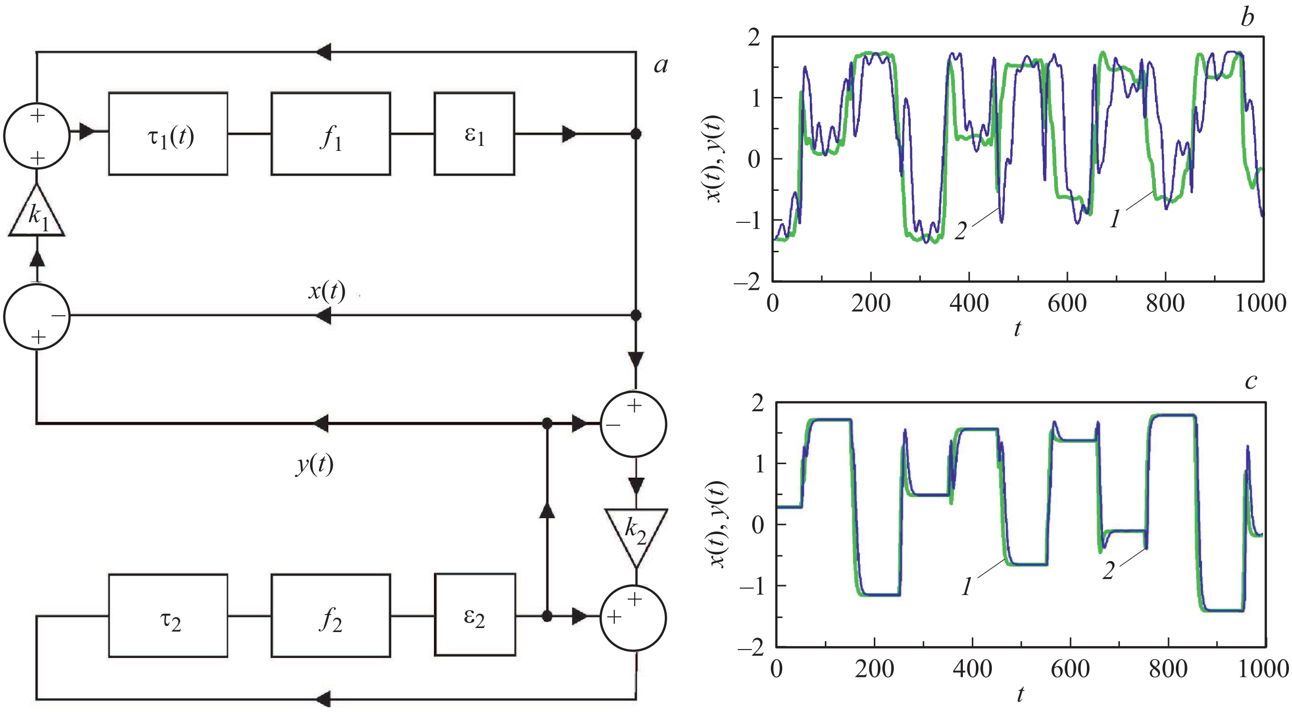

Figure 2. *a* — block diagram of two mutually coupled time-delay systems. Elements $\tau_1(t)$ and τ_2, f_1 and f_2, and ε_1 and ε_2 implement delay, nonlinear transformation, and inertial transformation of oscillations, respectively. Elements k_1 and k_2 define the strength of coupling between the systems. *b* and *c* — time series of $x(t)$ (*1*) and $y(t)$ (*2*) oscillations at $k_1 = k_2 = 0.1$ and 0.8, respectively.

oscillations at $k_1 = k_2 = 0.1$. The oscillation regimes in both systems at this coupling strength differ considerably from the regimes established without coupling. The case of strong coupling ($k_1 = k_2 = 0.8$) is presented in Fig. 2, *c*. Here, both systems manifest laminar chaos and are synchronized completely.

Thus, we have demonstrated for the first time that laminar chaos may emerge in a system with a constant delay time coupled with a system in the laminar chaos regime. Therefore, the regime of laminar chaos may be established both by modulating the delay time and by introducing coupling between time-delay systems.

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Conflict of interest

The authors declare that they have no conflict of interest.

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