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Helmholtz-Gauss beams with quadratic radial dependence

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A new class of localized solutions of paraxial parabolic equation is introduced. Each solution is a product of some Gaussian-type localized axisymmetric function (different from the fundamental mode) and an amplitude factor. The latter can be expressed via an arbitrary solution of the Helmholtz equation on an auxiliary two-sheet complex surface. The class under consideration contains both well known and novel solutions, including those describing optical vortices of various orders.

Keywords: parabolic equation, quadratic beams, Gauss, Helmholtz, Bessel.

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1. Introduction

In this article, we build a new class of solutions of the parabolic equation [1,2]

$$2iku_z + \Delta_\perp u = 0, \tag{1}$$

(also called the paraxial wave equation [3]), where $\Delta_\perp = \partial_{xx} + \partial_{yy}$ and $k = \text{const}$ is a wave number. It is assumed that $|u| \to 0$ for $|x| + |y| \to \infty$. We will call z the longitudinal coordinate, and x and y transverse coordinates.

Solutions of equation (1) are used for approximate description of time-harmonic wave propagation along axis z [1–4] when paraxiality conditions are fulfilled [3,4]. Solutions of equation (1) can also be used as a technical tool for building exact time-nonharmonic solutions of the wave equation for which fulfillment of paraxiality conditions is not required [4–9].

The class considered in the article is built on the basis of quadratic Bessel–Gaussian beams [7] and includes several subclasses, one of which is well-known astigmatic Gaussian beams, whose relation to quadratic Bessel–Gaussian beams is studied in [10], others are similar to asymmetric [11] and shifted [12] Bessel–Gaussian beams. Among them there are solutions that describe optical vortices of various orders, which opens up the prospect of their use in numerous applications from manipulation of microparticles to information transmission [13].

The beams belonging to the class under consideration mostly inherit the geometric properties of quadratic Bessel–Gauss beams, which distinguish them from the classical (linear) Bessel–Gauss beams [7,14,15]. As noted in [7], „while the latter have an essentially conical geometry, the former propagate colinearly“. This is due to the fact that quadratic Bessel–Gaussian beams are components of the Fourier series expansion in the angular variable of an astigmatic Gaussian beam propagating along the optical axis [10], and classical ones are that of axisymmetric Gaussian beam, but inclined to the optical axis and/or shifted in the transverse direction [15].

When building the class under consideration, a new technical tool is used that is a two-sheeted surface together with solutions of the Helmholtz equation on this surface. To avoid false associations, we want to emphasize that this surface is not a Riemann surface associated with any analytic function of a complex variable. An important role in our consideration is played by the secondary parabolic equation, which arose earlier in the article [16] when constructing a class of Helmholtz–Gauss solutions with linear radial dependence. In articles [17,18] this approach is generalized to media with quadratic dependence of the refractive index on the radius.

2. Laplace–Gauss and Helmholtz–Gauss solutions and a secondary parabolic equation

The fundamental mode of the equation (1) is a well known Gaussian beam [19], having the form

$$G = \frac{C}{q(z)} \exp\left\{\frac{ik}{2}\frac{r^2}{q(z)}\right\}, \tag{2}$$

where $r^2 = x^2 + y^2$, $q(z) = z - z_0 - ib$, z_0 and $b > 0$ are real constants, and C is a complex constant. This function is Gaussian localized in transverse coordinates. Solutions of equation (1) having the form

$$u = AG, \tag{3}$$

where $A = A(x, y, z) \neq \text{const}$, are called higher modes. We call the function A the amplitude [16]. Substituting (3) into (1), we obtain

$$2ikA_z + \Delta_\perp A + 2ik\,\frac{xA_x + yA_y}{q(z)} = 0.$$

Following [16,17,20], having performed a complex change of variables

$$X = \frac{x}{q(z)}, \quad Y = \frac{y}{q(z)}, \quad Z = -\frac{1}{q(z)}, \tag{4}$$

after some transformations, we arrive at a parabolic equation for the amplitude:

$$2ikA_Z + \widehat{\Delta}A = 0, \tag{5}$$

where

$$\widehat{\Delta} = \partial_{XX} + \partial_{YY}. \tag{6}$$

According to the terminology proposed in [17], equation (5) is called a secondary parabolic equation.

If the amplitude A does not depend on Z:

$$A = \Psi(X, Y),$$

we arrive at the Laplace equation

$$\widehat{\Delta}\Psi = 0$$

and get the Laplace–Gaussian modes [16,20]

$$u = \Psi\left(\frac{x}{q(z)}, \frac{y}{q(z)}\right) G,$$

where Ψ is an arbitrary harmonic function.

If the amplitude depends exponentially on Z:

$$A = \exp\left(-i\frac{K^2}{2k}Z\right)\Psi(X, Y),$$

where K is an arbitrary complex constant, then we arrive at the Helmholtz equation

$$\widehat{\Delta}\Psi + K^2\Psi = 0 \tag{7}$$

and get the Helmholtz–Gauss modes [16,20,21]

$$u = \Psi\left(\frac{x}{q(z)}, \frac{y}{q(z)}\right) \exp\left(\frac{iK^2}{2kq(z)}\right) G,$$

where Ψ is an arbitrary solution of equation (7).

Particular cases of such modes are the Bessel–Gauss beams built by Gori, Guattari and Padovani [7,14,15]

$$u_m = \frac{C}{q(z)} \exp\left\{\frac{ikr^2}{2q(z)} + \frac{iK^2}{2kq(z)} + im\phi\right\} J_m\left(\frac{Kr}{q(z)}\right), \tag{8}$$

where ϕ is the polar angle on the xy plane, and their generalizations [11,12,17,22].

Note that the values of complex coordinates X, Y corresponding to points of physical space are not completely independent: as can be seen from (4), their arguments either coincide or differ by π, i.e. X and Y are linearly dependent over the real number field. The set of such pairs (X, Y), which we will call a physical sheet, can be parametrized in terms of polar coordinates:

$$X = R\cos\Phi, \quad Y = R\sin\Phi, \tag{9}$$

where $R = r/q(z)$ is the complex polar radius and $\Phi = \phi$ is the real polar angle. The Laplace operator on a physical sheet, expressed in terms of R, Φ, has the form

$$\widehat{\Delta} = \frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \Phi^2}, \tag{10}$$

in this case, the 2π-periodicity condition is assumed to be satisfied for the variable Φ. In particular, Bessel–Gauss beams (8) can be obtained by selecting a solution of equation (7), which has the form

$$H = J_m(KR)\exp(im\Phi).$$

The techniques presented in this section will be used in constructing solutions of another type — quadratic Helmholtz–Gauss beams.

3. Quadratic Bessel–Gauss and Helmholtz–Gauss beams

Let us consider another type of localized solutions of equation (1), namely, Bessel–Gauss beams with quadratic radial dependence or, briefly, quadratic Bessel–Gauss beams, found by Caron and Potvlidge [7]. These solutions in the article [7] are presented in the form

$$u_m = E_m \frac{w_0}{W(z)} \exp\left[-\left(1 + i(\mu^2+1)\frac{z}{z_R}\right)\frac{r^2}{W^2(z)}\right] \times J_{|m|/2}\left[\frac{\mu r^2}{W^2(z)}\right]\exp(im\phi), \tag{11}$$

where $J_{|m|/2}$ is the Bessel function of $|m|/2$ order, and

$$W(z) = w_0\sqrt{1 - (\mu^2+1)\left(\frac{z}{z_R}\right)^2 + 2i\frac{z}{z_R}}. \tag{12}$$

In this case w_0 and

$$z_R = \frac{kw_0^2}{2}$$

are real, and E_m and μ are complex parameters characterizing the solution. The beam (11) is Gaussian localized in transverse coordinates for $|\mathrm{Im}\mu| < 1$. Such functions are essentially different from (8). In particular, the argument of the Bessel function contains r^2 instead of r, and its index is $|m|/2$. At the same time, the vortex on the beam optical axis (11) has the same topological charge m as (8).

For further investigations, it will be convenient to use not the original form (11), which we will call *Scottish*, but the alternative (*Spanish*) form [10,23] proposed by Chamorro-Posada:

$$u_m = \frac{C}{\sqrt{q_1(z)q_2(z)}} \exp\left\{ik\xi(z)r^2 + im\phi\right\} J_{|m|/2}(k\eta(z)r^2), \tag{14}$$

where $q_j(z) = z - z_j - ib_j$, $j = 1, 2, z_j$ and $b_j > 0$ are real constants and C is a complex constant,

$$\xi(z) = \left(q_1^{-1}(z) + q_2^{-1}(z)\right)/4,$$

$$\eta(z) = \left(q_1^{-1}(z) - q_2^{-1}(z)\right)/4.$$

In the article [10], the equivalence, up to a shift over the longitudinal variable z, of the Scottish (11) and Spanish (14) representations of the quadratic Bessel–Gaussian beam was proved, and there was established link between the parameters characterizing these representations.

Note that the function

$$\widehat{G} = \frac{C}{\sqrt{q_1(z)q_2(z)}} \exp\left\{ik\xi(z)r^2\right\}, \tag{15}$$

unlike (2), does not satisfy equation (1). Nevertheless, we will look for solutions of equation (1) that generalize quadratic Bessel–Gauss beams (14) in a form similar to (3):

$$u = A\widehat{G}, \tag{16}$$

with some non-constant amplitude function $A = A(r, \phi, z)$. If we substitute (16) into (1) and perform a complex change of variables

$$R = \eta(z)r^2, \quad \Phi = 2\phi, \quad Z = \ln\frac{q_1(z)}{q_2(z)}, \tag{17}$$

then after some algebra (see Appendix) we arrive at a secondary parabolic equation for the amplitude, which has the form

$$2ikA_Z + R(\widehat{\Delta} + k^2)A = 0, \tag{18}$$

with periodic conditions

$$A(Z, R, \Phi + 4\pi) = A(Z, R, \Phi), \tag{19}$$

arising from the requirement of uniqueness of solution in the physical space. In equation (18), the analytical expression for the operator $\widehat{\Delta}$ coincides with (10). The difference from the case considered above lies in the non-standard periodic conditions (19) with respect to the variable Φ, which plays the role of an angle. Therefore, we will now interpret the operator $\widehat{\Delta}$ in equation (18) as a Laplacian on an auxiliary two-sheeted complex surface with a branch point at $R = 0$, the first sheet of which corresponds to $\Phi \in [0, 2\pi)$, and the second one to $\Phi \in [2\pi, 4\pi)$ with cuts at $\Phi = 2\pi n$. Each of these sheets is similar to the physical sheet considered above, which arises when constructing the Laplace–Gauss and Helmholtz–Gauss modes with the usual (linear) radial dependence [16,20,21]. As before, the points of such a surface are characterized by a complex radial variable R and a real angular variable Φ.

We confine ourselves to considering solutions (18) that do not depend on Z:

$$A = \Psi(R, \Phi). \tag{20}$$

In this case we arrive at the Helmholtz equation

$$(\widehat{\Delta} + k^2)\Psi(R, \Phi) = 0. \tag{21}$$

We see that an arbitrary solution $\Psi(R, \Phi)$ of the Helmholtz equation on a two-sheeted surface corresponds to some solution of equation (1) in the original physical space:

$$u = \Psi(\eta(z)r^2, 2\phi)\widehat{G}. \tag{22}$$

If the first cofactor is limited or does not grow too fast, then the function (22) is localized with respect to transverse coordinates. In this case, it is natural to call such a solution of equation (1) a quadratic Helmholtz–Gauss beam. We emphasize that if Ψ is 4π-periodic over Φ, then the function u is 2π-periodic over ϕ respectively.

An arbitrary solution of equation (21) on a two-sheeted surface can be represented as a sum of two solutions, one of which is 2π-periodic over Φ, and the other is 2π-antiperiodic. In physical space, this corresponds to even and odd solutions with respect to the rotation through the angle π around the optical axis. The first term is a smooth solution of equation (21) on an one-sheet surface (i.e., on a plane with a complexified radial variable), and the second term on the first sheet ($\Phi \in [0, 2\pi)$) satisfies equation (21) with boundary conditions on the cut:

$$\begin{cases} \Psi(R, 2\pi) = -\Psi(R, 0), \\ \Psi_\Phi(R, 2\pi) = -\Psi_\Phi(R, 0), \end{cases}$$

and in this case, it continues to the second sheet ($\Phi \in [2\pi, 4\pi)$) in an odd way.

4. Examples

4.1. Quadratic Bessel–Gauss beams

If we take

$$\Psi_m = J_{|m|/2}(kR)\exp\left\{i\frac{m}{2}\Phi\right\} = J_{|m|/2}(k\eta(z)r^2)\exp\{im\phi\}, \tag{23}$$

then the function (22) will coincide with (14).

4.2. Asymmetric quadratic Bessel–Gauss beams

The constructions of this section are based on an approach that made it possible to obtain asymmetric Bessel modes in the article [24] and asymmetric Bessel–Gaussian beams in the article [11]. We use the identity 5.7.6.1 from [25]

$$\sum_{p=0}^{\infty} \frac{t^p}{p!} J_{p+\nu}(s) = s^{\nu/2}(s-2t)^{-\nu/2} J_\nu\left(\sqrt{s(s-2t)}\right), \quad (24)$$

$$|2t| < s.$$

In the case $s = kR$, $\nu = |m|/2$, $t = 2kc\exp(i\Phi\kappa)$ ($\kappa = 1$ for $m \geq 0$, $\kappa = -1$ for $m \leq 0$, for $m = 0$ any sign can be taken, c is some complex constant) it follows from (24) that

$$\tilde{\Psi}_m = \sum_{p=0}^{\infty} \frac{(kc)^p}{p!} \Psi_{m+2p\kappa} = \left(\frac{R}{R - 2c\exp(i\Phi\kappa)}\right)^{|m|/4}$$

$$\times J_{|m|/2}\left(k\sqrt{R(R-2c\exp(i\Phi\kappa))}\right)\exp\left\{i\,\frac{m}{2}\,\Phi\right\}. \quad (25)$$

These solutions (asymmetric Bessel modes) were found in [24] for the case of even positive m when they are regular on an one-sheet surface, and were used in [11] to construct asymmetric Bessel-Gauss beams. In the general case, despite the restriction $|2t| < s$ (i.e., $|2c| < R$) in formula (24), function (25) satisfies the Helmholtz equation (21) on the entire two-sheeted surface for an arbitrary value c, which can be verified by direct calculation. In this case, at the point where the difference $R - 2c\exp(i\Phi\kappa)$ vanishes, the function (25) has a removable singularity. For $c = 0$ the functions (25) coincide with (23). If we return to the original variables, then (25) takes the form

$$\tilde{\Psi}_m = \left(\frac{\eta(z)r^2}{\eta(z)r^2 - 2c\exp(2i\phi\kappa)}\right)^{|m|/4} \exp\{im\phi\}$$

$$\times J_{|m|/2}\left(kr\sqrt{\eta(z)(\eta(z)r^2 - 2c\exp(2i\phi\kappa))}\right). \quad (26)$$

The product (22) gives a new family of localized solutions of equation (1), which, by analogy with [11], we call asymmetric quadratic Bessel–Gauss beams [26]. Such solutions contain an optical vortex with a topological charge m located on the optical axis, and vortices with a topological charges ± 1, the locations of which are determined by the values of the Bessel function roots.

4.3. Singular quadratic Bessel–Gauss beams and their regularization

Let us consider the functions

$$\Psi_m^- = J_{-|m|/2}(kR)\exp\left\{i\,\frac{m}{2}\,\Phi\right\}. \quad (27)$$

Such functions for $R > 0$ satisfy the Helmholtz equation (21) on a two-sheeted surface. For even m, functions (27) either coincide with (23) or differ from them in sign and do not give new solutions to equation (1). If the values of m are odd, the functions (27) can no longer be expressed in terms of (23), and the Bessel functions with negative semi-integer indices grow infinitely at $R \to 0$. Accordingly, the functions (22) — singular symmetric Bessel–Gauss beams— have a singularity on the optical axis and, apparently, have no direct physical meaning.

Nevertheless, let us use formula (24) and construct new asymmetric singular solutions of equation (21):

$$\tilde{\Psi}_m^- = \sum_{p=0}^{\infty} \frac{(kc)^p}{p!} J_{-|m|/2+p}(kR)\exp\left\{i\left(\frac{m}{2} - p\kappa\right)\Phi\right\}$$

$$= \sum_{p=0}^{[|m|/2]} \frac{(kc)^p}{p!} \Psi_{m-2p\kappa}^- + \sum_{p=1+[|m|/2]}^{\infty} \frac{(kc)^p}{p!} \Psi_{-m+2p\kappa}$$

$$= \left(\frac{R - 2c\exp(-i\Phi\kappa)}{R}\right)^{|m|/4}$$

$$\times J_{-|m|/2}\left(k\sqrt{R(R-2c\exp(-i\Phi\kappa))}\right)\exp\left\{i\frac{m}{2}\Phi\right\}, \quad (28)$$

$\kappa = \mathrm{sign} m$, square brackets indicate the integer part of the number. The respective functions (22) with singularities on the optical axis will be called singular asymmetric Bessel–Gaussian beams.

Let's remove the singularities on the axis, for which we compose a linear combination of the obtained functions, which will be bounded at $R \to 0$. The most obvious way is to consider the series (28) again, excluding the singular terms from it and leaving only the regular ones:

$$\Psi = \tilde{\Psi}_m^- - \sum_{p=0}^{[|m|/2]} \frac{(kc)^p}{p!} \Psi_{m-2p\kappa}^- = \sum_{p=1+[|m|/2]}^{\infty} \frac{(kc)^p}{p!} \Psi_{-m+2p\kappa}$$

$$= \left(\frac{R - 2c\exp(-i\Phi\kappa)}{R}\right)^{|m|/4}$$

$$\times J_{-|m|/2}\left(k\sqrt{R(R-2c\exp(-i\Phi\kappa))}\right)\exp\left\{i\frac{m}{2}\Phi\right\}$$

$$- \sum_{p=0}^{[|m|/2]} \frac{(kc)^p}{p!} J_{-|m|/2+p}(kR)\exp\left\{i\left(\frac{m}{2} - p\kappa\right)\Phi\right\}. \quad (29)$$

Such regularized amplitudes are bounded and tend to zero at $R \to 0$.

Let us describe another way of regularizing the solutions under consideration. Let $N > 1 + m/2$ be a natural number and $\omega_N = \exp(2\pi i/N)$. Let us consider N functions $\tilde{\Psi}_{m,j}^-$, $j = 1, \ldots, N$ of the form (28) differing in the values of the constants $c = c_j$, with $c_j = c_1\omega_N^{j-1}$. Then, as is easy to see, the linear combination

$$\Psi = \sum_{j=1}^{N} \omega_N^{j-1} \tilde{\Psi}_{m,j}^- \quad (30)$$

will be bounded at $R \to 0$, since the coefficients for all irregular terms will vanish.

Having calculated the product (22), we find a regularized quadratic Bessel–Gauss beam, a localized solution of equation (1). Regardless of the regularization method, all such solutions are odd with respect to rotation through the angle π around the optical axis, vanish on this axis, and have an optical vortex on it with an odd topological charge.

Note that since Bessel functions with semi-integer indices are expressible in terms of elementary functions [27], singular and regularized quadratic Bessel–Gaussian beams also have this property.

4.4. Quadratic Cosine–Gaussian Beams

Let us consider the simplest particular case of singular amplitude (27) corresponding to $m = \pm 1$ [27]:

$$\Psi^{-}_{\pm 1} = J_{-1/2}(kR)\exp\left\{\pm\frac{i\Phi}{2}\right\} = \sqrt{\frac{2}{\pi kR}}\cos kR\exp\left\{\pm\frac{i\Phi}{2}\right\}. \quad (31)$$

Then, according to (28),

$$\tilde{\Psi}^{-}_{\pm 1} = \sqrt[4]{\frac{R - 2c\exp(\mp i\Phi)}{R}} \times J_{-1/2}\left(k\sqrt{R(R-2c\exp(\mp i\Phi))}\right)\exp\left\{\pm\frac{i\Phi}{2}\right\} = \sqrt{\frac{2}{\pi kR}}\cos\left(k\sqrt{R(R-2c\exp(\mp i\Phi))}\right)\exp\left\{\pm\frac{i\Phi}{2}\right\}. \quad (32)$$

Functions (31) and (32) are unbounded at $R \to 0$ solutions of equation (21) on a two-sheeted surface.

Let us consider the regularization of such amplitudes. Since for $m = \pm 1$ series (28) contains only one singular term, the simplest regularized amplitude corresponding to (29) is just the difference $\tilde{\Psi}^{-}_{\pm 1} - \Psi^{-}_{\pm 1}$. In the general case, let $\tilde{\Psi}^{-}_{\pm 1, j}$, $j = 1, \ldots, N$ be functions of the form (32) differing in the values of the complex constants $c = c_j$, and now these constants are already arbitrary. Then the linear combination

$$\Psi = \sum_{j=1}^{N} C_j \tilde{\Psi}^{-}_{\pm 1, j} \quad (33)$$

in case of meeting the condition

$$\sum_{j=1}^{N} C_j = 0$$

is bounded and tends to zero at $R \to 0$. In the initial variables, the function (33) has the form

$$\Psi = \frac{\exp\{\pm i\phi\}}{r}\sqrt{\frac{2}{\pi k\eta(z)}} \times \sum_{j=1}^{N} C_j \cos\left(kr\sqrt{\eta(z)(\eta(z)r^2 - 2c_j\exp(\mp 2i\phi))}\right). \quad (35)$$

Then, if (34) is fulfilled, the product (22) is a regular localized solution of equation (1), odd with respect to rotation through the angle π about the optical axis and vanishing on this axis. Functions of this kind were called quadratic Cosine–Gaussian beams in [28].

4.5. Shifted quadratic Bessel–Gauss beams

We begin description of another solutions family by considering the Cartesian coordinates on the two-sheeted surface under consideration:

$$X = R\cos\Phi = \eta(z)(x^2 - y^2), \quad (36)$$

$$Y = R\sin\Phi = 2\eta(z)xy. \quad (37)$$

Obviously, changing Φ by 2π does not change the values of X and Y. Therefore, any analytic function that can be expressed in terms of (36) and (37) is 2π-periodic over Φ and, therefore, π-periodic over ϕ. For such functions, the operator $\widehat{\Delta}$ is defined by formula (6).

Notice, that

$$R^2 = X^2 + Y^2 = (X + iY)(X - iY), \quad (38)$$

$$\exp\{i\Phi\} = \sqrt{(X + iY)/(X - iY)} = (X + iY)/R = R/(X - iY). \quad (39)$$

With the help of (38) and (39) we can transform even-numbered functions (23) regular on an one-sheet surface:

$$\Psi_{2m} = J_{|m|}(kR)\exp\{im\Phi\} = J_{|m|}\left(k\sqrt{X^2 + Y^2}\right)\left(\frac{X + iY}{X - iY}\right)^{m/2}. \quad (40)$$

Let us now perform a shift in (40) by a constant complex vector (X_0, Y_0): $X \mapsto X' = X - X_0$, $Y \mapsto Y' = Y - Y_0$ and obtain new solutions of the Helmholtz equation (21):

$$\Psi'_{2m} = J_{|m|}(kR')\left(\frac{X' + iY'}{X' - iY'}\right)^{m/2}, \quad (41)$$

where

$$R' = \sqrt{X'^2 + Y'^2} = \sqrt{(X' + iY')(X' - iY')}.$$

Such solutions of the Helmholtz equation were considered in [29] and used to build shifted Bessel-Gaussian beams in [12,17].

Using (41) as the amplitude function in (22), we obtain a new family of solutions of the parabolic equation (1). Notice that in the case of $X_0 + iY_0 = 0$ (for $m > 0$) or $X_0 - iY_0 = 0$ (for $m < 0$), (41) turns into (25) for some c, and, consequently, the shifted quadratic Bessel-Gaussian beams turn into the asymmetric ones considered earlier.

Note that an optical vortex with a topological charge m, located on the optical axis in case of an non-shifted quadratic Bessel–Gauss beam, as a result of a complex shift, generally speaking, changes its position and ends up at the point where $X' + iY'$ vanishes for $m > 0$ or $X' - iY'$ vanishes at $m < 0$.

4.6. Astigmatic Gaussian beams

The class of quadratic Helmholtz–Gaussian beams turns out to contain the well-known astigmatic Gaussian beams. In particular, if we take the simplest solution of the Helmholtz equation (21), which has the form of a plane wave on an auxiliary surface:

$$\Psi = \exp(ikX) = \exp\big(ik\eta(z)(x^2 - y^2)\big), \qquad (42)$$

product (22) takes the form of aligned simple astigmatic (ASA) Gaussian beams [19]:

$$u = \frac{C}{\sqrt{q_1(z)q_2(z)}} \exp\left\{\frac{ik}{2}\left[\frac{x^2}{q_1(z)} + \frac{y^2}{q_2(z)}\right]\right\}.$$

Having selected another wave direction on the (XY) plane:

$$\Psi = \exp\big(ik(X\cos\varphi + Y\sin\varphi)\big), \qquad (43)$$

where φ — a real constant, we arrive at rotated simple astigmatic (RSA) Gaussian beams [19] at $\varphi \neq n\pi$, $n \in \mathbb{Z}$. If we take the nonreal value φ in (43), we obtain, under certain restrictions on Imφ, the general astigmatic (GA) Arnaud–Kogelnik beam [8,19,30].

It is interesting to note that, as shown in the article [10], when expanding a Gaussian beam with simple astigmatism or a general astigmatic Arno–Kogelnik beam into a Fourier series over the variable ϕ, the terms of such an expansion have the form of quadratic Bessel–Gauss beams with even numbers.

5. Conclusion

In this article quadratic Helmholtz–Gauss modes are constructed, which are a new class of localized solutions of a paraxial parabolic equation. This class contains both well-known solutions, such as quadratic Bessel–Gaussian beams and astigmatic Gaussian beams, as well as new families of solutions that require detailed study.

The constructed class contains both paraxial and non-paraxial solutions of equation (1). Paraxial solutions can be used for an approximate description of time-harmonic wave propagation along the z axis. At the same time, paraxiality is not required to build exact nonstationary solutions of the wave equation.

We would like to note that the secondary parabolic equation obtained in this article, in our opinion, can be used to build not only the Helmholtz–Gaussian modes, but also other types of solutions.

We also would like to note that, apparently, the possibility of the existence of Helmholtz–Gauss beam classes with a dependence on the radius that is different from linear and quadratic seems quite possible. This idea is suggested by the solutions found in the article [31], whose structure resembles asymmetric Bessel–Gaussian beams, in which, however, the argument of the Bessel function at large distances from the optical axis is of the order of $r^{3/2}$.

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Conflict of interest

The authors declare that they have no conflict of interest.

Appendix

Derivation of the secondary parabolic equation (18)

After substitution (16) into equation (1) we obtain the equation for the amplitude:

$$2ikA_z + \frac{1}{r}\partial_r(rA_r) + \frac{1}{r^2}A_{\phi\phi}4ik\xi(z)rA_r$$
$$+ 4k^2\eta^2(z)r^2A = 0. \qquad (44)$$

Let us perform a complex change of variables by introducing new coordinates (17). In these variables

$$A_z = 4\eta A_Z - 4\xi R A_R,$$
$$\frac{1}{r}\partial_r(rA_r) = 4v(A_R + RA_{RR}),$$
$$4ik\xi rA_r = 8ik\xi RA_R,$$
$$\frac{1}{r^2}A_{\phi\phi} = \frac{4v}{R}A_{\Phi\Phi},$$

and equation (44) takes the form

$$8ikvA_Z + 4vR\left(\frac{A_R}{R} + A_{RR} + \frac{1}{R^2}A_{\Phi\Phi} + k^2A\right) = 0,$$

which after reductions coincides with (18).

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