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Shifting the flow stability limit in the presence of random fluctuation of the rotational velocity

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Nonlinear stability of the isothermal three-dimensional flows of viscous incompressible fluid in the rotational spherical layer with the noise presence was studied numerically. Transition between stable non-stationary flow and unstable flow in the form of travelling azimuthal waves is under consideration. Small-amplitude noise inserted to the flow by random broadband disturbances of the inner sphere rotational rate about constant at time averaged value, outer sphere is fixed. An approach is proposed to simplify finding the critical Reynolds number, corresponding to the stability limit in the presence of noise. The results obtained by the proposed and well known methods are compared.

Keywords: noise, rotational flows, spherical Couette flow, instability control.

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Introduction

The large-scale atmospheric flows are exposed to external effects, which are different in a spectrum type and have a wide range of time scales — from modulation of solar radiation coming to the atmosphere [1] to the wide-band millisecond changes of the day duration [2]. These effects are regarded as a noise external to the flow, whose background forms coherent structures as a result of instability development. The critical Reynolds number Re_c corresponding to a stability limit can change both under common influence of the noise and the periodic modulation [3] and by impact of the external noise only [4]. Possibly, it is correlated to generation of middle flows [4–6]. The noise increases increments of rise of the linear modes and the interaction time of the modes and can also affect selection of a wave number of the azimuth mode [7]. The noise can result in changing the frequency and diffusion of the oscillation phase [8,9] as well as to flow chaotization [10,11]. In rotational flows, the stability loss and chaotization can depend not only on the noise level, but on its time characteristics [12].

For the rotational spherical flows being investigated in the present study, it has been numerically demonstrated by axis-symmetrical formulation in [4] that under the noise impact the kinetic energy of the flows increases, while the stability limit decreases in proportion to the increase in the noise level. These results have been obtained in the simplified formulation which regarded the linear stability of the velocity field averaged across the large time interval, wherein the instantaneous velocity fields were calculated at random wide-band fluctuations of the rotational rate of the inner sphere. However, it is unclear whether solutions of these problems with various types of the external effects including with the noise presence are unique or not [13].

In this regard, it is more and more important to analyze the flow stability using the full three-dimensional Navier–Stokes equations. The critical Reynolds number Re_{c0} corresponding to the stability limit without the noise is assumed to be determined by approximation of the dependence of the oscillation amplitude on supercriticality $A \sim (\mathrm{Re} - \mathrm{Re}_{c0})^{1/2}$ at $(\mathrm{Re} - \mathrm{Re}_{c0}) \to 0$, here A —a calculated amplitude of the velocity oscillations after the flow losses the stability and travelling azimuth waves are formed [14]. However, the results of flow calculation with the two-dimensional convection [3] have shown that the above-said approximation can not be applied in the presence of noise. In this regard, it is necessary to perform the calculations within the area $(\mathrm{Re} - \mathrm{Re}_c) \to 0$. But, as it will be shown, the substantial increase in the time of transients within this area does not allow reliably calculating small values A. That is why it is desirable to find the approximation of the dependence of A on Re in this case, too. The purpose of this study is, when there are the random fluctuations of the rotational rate, to numerically determine the value Re_c by finding the dependence of A on Re based on solving the full three-dimensional system of the Navier–Stokes equations, as well as to show advantages of the new approach determining Re_c with less calculation efforts.

1. Calculation method and study region

It investigates isothermal flows of the viscous incompressible unstratified fluid in the rotational spherical layer, which are described in the inertial system of coordinates by the Navier–Stokes and continuity equations:

$$\frac{\partial U}{\partial t} = U \times \mathrm{rot}\, U - \mathrm{grad}\left(\frac{p}{\rho} + \frac{U^2}{2}\right) - \nu \mathrm{rotrot}\, U, \, \mathrm{div} U = 0.$$

Here U, p — the field of the flow velocity and pressure, respectively; ρ, ν — the density and kinematic viscosity of the fluid. The boundary conditions include adhesion and non-permeability, which in the spherical system of coordinates with the radial (r), polar (θ) and azimuth (φ) directions can be presented as follows:

$$u_\varphi(r = r_k) = \Omega_k(t) r_k \sin\theta, \;\; u_r(r = r_k) = 0,$$

$$u_\theta(r = r_k) = 0, \;\; k = 1, 2.$$

Here u_φ, u_r, u_θ — the azimuth, radial and polar components of the velocity U, the indices 1 and 2 relate to the internal ($k = 1$) and external ($k = 2$) solid spherical boundaries; Ω_k, r_k — the angular rotational rate and the radius of the corresponding sphere. As noted above, the present study has a fixed external sphere, $\Omega_2 = 0$, and in this regard the subscript in Ω_1 and Re_1 is omitted. Additional noise is contributed to the flows, in the same way as in [4,7], by the random wide-band fluctuations of the rotational speed of the inner sphere:

$$\Omega(t) = \Omega_0 + Nrn(j).$$

Here Ω_0 — the averaged part of the rotational rate, which is constant in time, $rn(j)$ — the pseudo-random number from the sequence with the standard normal distribution and averaged zero value, which is changed in each step in time, N — the noise amplitude. As in [4,7], the fluctuation spectrum of the rotational rate has the same amplitude within the wide frequency range and is the „white noise“. The noise amplitude is determined by the expression

$$N = \frac{1}{\Omega_0}\sqrt{\frac{1}{K-1}\sum_{i=1}^{K}\big(\Omega(t_i) - \Omega_0\big)^2},$$

where K — the length of the time sampling. All the calculations are performed at $N = 0.04$ to compare with the results of the study [4], which has analyzed the linear stability in the axis-symmetrical simplified formulation (see above) for the same value N. As in [4,15], the same sequence of the random numbers has been used in all the calculations.

The numerical solution has included the algorithm [16] and the programs [16,17] based thereon, with a conservative finite-difference scheme of discretization of the Navier–Stokes equations (NSE) across the space and with a semi-implicit Runge–Kutta scheme of the third accuracy order for time integration. In contrast to [4], the calculated non-stationary flows have no condition of symmetry relative to the equator plant and rotation axis. The space discretization has been performed by decreasing a mesh size near the spheres (along r) and the equator plane (along θ). The ratio of the maximum mesh size to the minimum one was two, while the total number of nodes was $5.76 \cdot 10^5$. The dependence of the calculation results on a configuration of the computational grid has been investigated in detail in many studies (for example, in [18]). The calculations have been performed at size parameters, which correspond to conditions of the experiments and calculations in [4]: $\nu = 5 \cdot 10^{-5}\,\mathrm{m^2/s}$, $r_1 = 0.075\,\mathrm{m}$, $r_2 = 0.15\,\mathrm{m}$, and with the constant time step of $dt = 0.025\,\mathrm{s}$. The calculations have been performed with recording the flow velocities in time in points within the meridional flow plane at rays with angular declinations from the rotational axis $\theta = 17.84^\circ$ (near the rotation axis), 45.63° (in the middle latitudes) and $\theta = 89.67^\circ$ (near the equator plane) and relative distances to the inner sphere $R = 0.177$, 0.284, 0.381, 0.487, 0.60, 0.67 and 0.76, where $R = (r - r_1)/(r_2 - r_1)$. The point with the coordinates $R = 0.67$ and $\theta = 45.63^\circ$ has been selected to present the results as being the closest to a position of the point of measuring the azimuth component of the velocity in the experiment [4]. The calculation results are given below in dependence on the Reynolds number $\mathrm{Re} = \Omega r_1^2/\nu$, the calculation have been performed within the range of changing Re from 488 to 500.

2. Results

Let us recall that the present study investigates noise impact on changing the position of the flow stability limit in the rotational spherical layer. Without noise, the flow prior to the stability loss is stationary and symmetrical relative to the rotation axis and the equator plane [19]. When adding the random fluctuations to a rotational rate signal, the above-said symmetries are preserved, but the flow velocity field becomes random and non-stationary [4]. Let us consider how the random fluctuations of the rotational speed of the inner sphere propagate within the flow prior to their stability loss. This process can be illustrated by mean-square deviations of the azimuth component of the speed rms u_φ (Fig. 1):

$$\mathrm{rms}\, u_\varphi(r, \theta) = \sqrt{\frac{1}{K_t}\sum_{k=1}^{K_t}\big(u_\varphi(t_k, r, \theta) - u_{\varphi av}(r, \theta)\big)^2}.$$

Here, the left summand is an instantaneous azimuth velocity at the moment of time t, while the right summand is an azimuth flow velocity averaged in time. If in the equator plane the values rms u_φ (the curve *2* of Fig. 1, *a*) exponentially decay radially with increased distance from the internal spherical boundary, in the middle latitudes the local maximum is evident near $R = 0.5$ (the curve *4* of Fig. 1, *a*). For comparison, Fig. 1, *a* also shows the values rms u_φ (the curves *1, 3*) at the periodic fluctuations of the rotational rate of the inner sphere: $\Omega(t) = \Omega_0(1 + A\sin(2\pi g t + \varphi))$, A, g — the amplitude and the frequency of the modulation, Ω_0 — the averaged rotational rate of the inner sphere. The case of Fig. 1 with $g = 0.3\,\mathrm{Hz}$ has evident satisfactory qualitative compliance of propagation of the random and periodic oscillations. In case of changing the value g the qualitative compliance is kept in the equatorial plane only. Figure 1, *b* shows the values rms u_φ, which are normalized

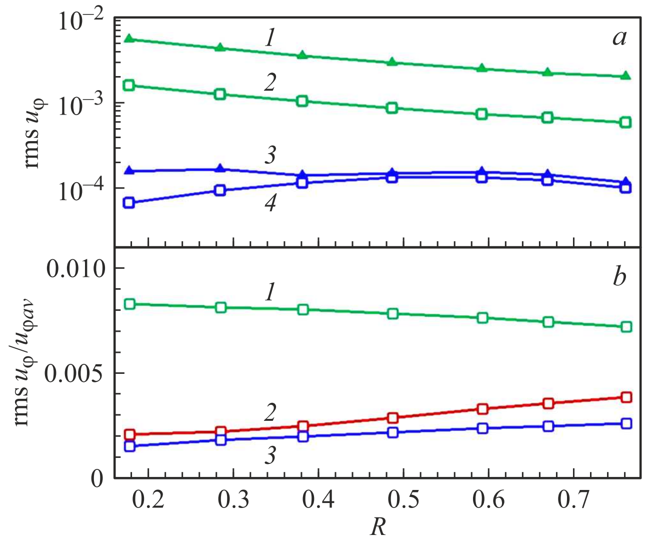

Figure 1. Dependence of rms u_φ, [m/s] (a) and rms $u_\varphi/u_{\varphi av}$ (b) on R at $\mathrm{Re} = 488$. a — *1, 3* — the periodic oscillations of the rotational rate with $g = 0.3\,\mathrm{Hz}$, $A = 0.046$; *2, 4* — the random fluctuations of the rotational rate $N = 0.04$, *1, 2* — $\theta = 89.67°$, *3, 4* — $\theta = 45.63°$. b — the random fluctuations of the rotational rate, θ: *1* — 89.67°, *2* — 17.84°, *3* — 45.63°.

per a value of the average azimuth component of the flow velocity $u_{\varphi av}$. If in the equator plane rms $u_\varphi/u_{\varphi av}$ decrease with increased distance from the internal sphere (the curve *1*), then in the middle latitudes and near the rotation axis (the curves *2* and *3*) there is evident increase in rms $u_\varphi/u_{\varphi av}$ with increase in R, which is correlated to decrease in $u_{\varphi av}$ when approaching the external boundary.

In the studied spherical layer with the relative thickness $\delta = (r_2 - r_1)/r_1 = 1$, at the stationary boundary conditions the flow losses the stability at $\mathrm{Re} > \mathrm{Re}_{c0}$ softly, without hysteresis [19,20]. At the same time, it forms travelling azimuth waves [19] (propagating along the rotation direction of the inner sphere) with the wave numbers $m = 3$ and/or $m = 4$. All the velocity components change in time periodically within the whole layer [19,20]. Both without noise [19,20], as well as with its presence [4], a dominant mode is $m = 4$, as both the cases comply with the relationship $\lambda_4 > \lambda_3$, where λ_4 and λ_3 — the rise increments of the linear modes with the wave numbers $m = 4$ and 3, respectively. It is the mode $m = 4$ that is taken to perform all further calculations.

In the presence of noise, the flows have been calculated before and after stability loss. The Fig. 2 shows time dependences u_φ at the middle latitudes ($\theta = 45.63°$). It is clear that the nature of random fluctuations of the velocity changes in the non-stationary flow prior to the stability loss (the curves *3* and *4* of Fig. 2) with increase in R, i.e. with increased distance from the noise source. As it gets closer to the internal sphere (the curve *3*), there are evident low-amplitude high-frequency fluctuations, whereas a low-frequency signal of the high amplitude prevails near the external sphere (the curve *4*). It is correlated to sharper decay of the high-frequency components in comparison with the low-frequency one and fully corresponds to the previously obtained spectra of the fluctuations of the rotation rate of the inner sphere and the spectra of the azimuth components of the rate at the distance thereto (Fig. 1 in [4]). The flow velocity signal after the stability loss (the curves *1, 2* of Fig. 2) is dominated by the periodic time dependence with the average value and the amplitude fluctuating in time.

The calculations in the axis-symmetrical formulation of the flows prior to the stability loss [4] have shown that adding the noise as the fluctuations of the rotational rate of the internal sphere can result in changing an average velocity profile. The generation of the middle flows is well known and for other types of non-stationary boundary conditions and it is caused by available non-linearity and viscosity in UCN [6]. Figure 3 shows noise-caused change of the relative increments of the azimuth component of the velocity V_φ in the flows before and after the stability loss, and also compares it with the case of periodic oscillations of the rotational rate ($V_\varphi = (u_{\varphi av} - u_{\varphi 0})/u_{\varphi 0}$, $u_{\varphi 0}$ — the azimuth component of the rate at the stationary boundary conditions). In the flows both before and after the stability loss, $V_\varphi > 0$, which is indicative of the generation of the middle flow along the azimuth direction. Near the rotation axis and at the middle latitudes, V_φ increases radially, and the same dependence R is typical for the periodic oscillations of the rotational rate. Reduction of V_φ with increased distance from the inner sphere is observed only in the equatorial plane. The biggest relative increments of the azimuth component of the rate are observed in a flow area near the rotation axis and the external sphere. The same result has been previously obtained for the axis-symmetrical flows prior to the stability loss [4]. The values of V_φ usually are higher in the case of the flow prior to the stability loss (the solid lines of Fig. 3), with the exclusion of the area at the middle latitudes near the external sphere.

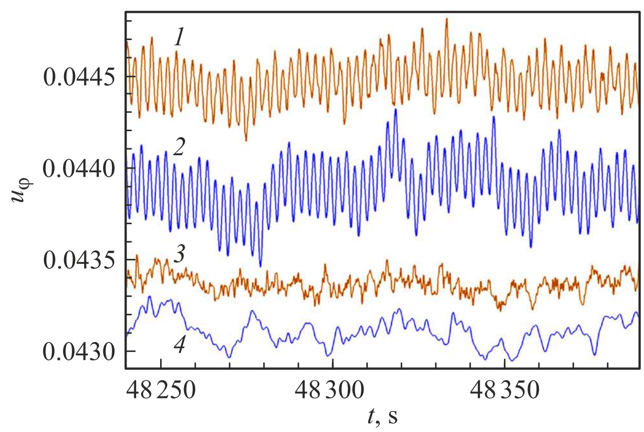

Figure 2. Dependence on the time t for the azimuth component of the speed u_φ [m/s] at $\mathrm{Re} = 492.77$ (*1, 2*) and 491 (*3, 4*); *1, 3* — $R = 0.177$, *2, 4* — $R = 0.76$. The dependences *2–4* are shifted upwards along the ordinate axis for clarity.

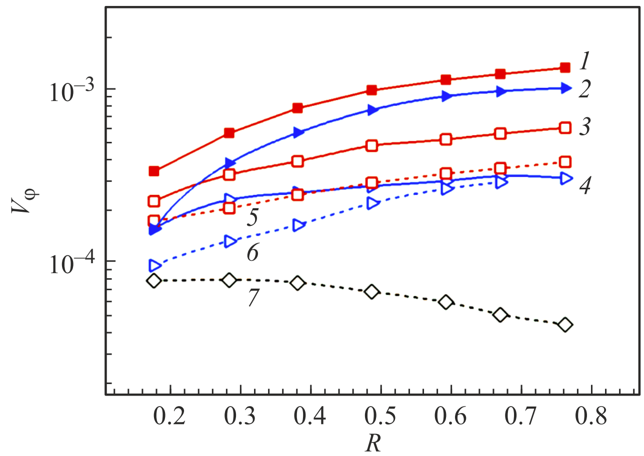

Figure 3. Dependence V_φ on the relative distance between the spherical boundaries R for the stable flows at $\mathrm{Re} = 488$, $g = 0.3$ Hz, $A = 0.046$ (*1, 2*, the solid lines, the close symbols), $\mathrm{Re} = 488$, the random fluctuations of the rotational rate (*3, 4*, the solid lines, the open symbols) and the unstable flows at $\mathrm{Re} = 492.9$ (*5–7*, the discontinuous lines, the open symbols). θ: *1, 3, 5* — 17.84°, *2, 4, 6* — 45.63°, *7* — 89.67°.

When determining the value of Re_c, the initial conditions can be taken to be beyond-critical Re values ($\mathrm{Re} > \mathrm{Re}_c$) [16]; after that the calculations are performed by reducing Re and in the traditional approach only at its beyond-critical values [16,18]. The present study investigates this approach by step-by-step reduction of Re from the selected initial value and by determining the amplitude of the oscillations A_f of the azimuth component of the velocity after the end of the transient and getting A_f to a time-constant value (let us recall that it is related to the case when $\mathrm{Re} > \mathrm{Re}_c$). At $\mathrm{Re} < \mathrm{Re}_c$, there is evident gradual decay of A_f without getting to the constant value.

The values of A_f, as in [7], were calculated using the Hilbert transform (HT). First of all, $u_\varphi(t)$ was filtered to have a signal $u_4(t)$ within the frequency range $f_4 \pm \Delta f$, which corresponded to the azimuth mode $m = 4$: $f_4 = 0.43$ Hz, $\Delta f = 0.005$ Hz. After that the value A_f was determined as a magnitude of the analytical signal:

$$A_f = \left| u_4(t) + \mathrm{i}\mathrm{HT}\big(u_4(t)\big) \right|.$$

In this method of determination of A_f it is possible to obtain the time dependence of the amplitude for the entire time sampling. It also included determination of the amplitudes of the azimuth modes in the experiments at $\mathrm{Re} < \mathrm{Re}_c$ [4].

Figure 4 shows the dependence of the value A_f on the Re number in the above-said flow point, in other flow regions the dependences are qualitatively the same. It is clear that when $\mathrm{Re} < 493$ the values A_f in the presence of noise exceed similar values corresponding to the stationary boundary conditions. And the stability limit determined by the equation $A_f = 0$ is shifted to lesser Re numbers in comparison with the case $N = 0$. This conclusion complies with results of the linear stability analysis for the time-averaged axis-symmetrical flows [4]. The shift of the stability limit due to noise can be determined by the Re-number dependence of not only A_f, but of other flow parameters, for example, rms u_φ (by the moment of getting the constant value, Fig. 5, *a*) or the normalized average value of the azimuth velocity $W = u_{\varphi av}/\Omega_0 r$ (by the kink point of the dependence on the Re number, Fig. 5, *b*).

Nevertheless, the very fact of shift of the stability limit position due to noise does not eliminate a problem of the value of this shift and accuracy of its determination. Without

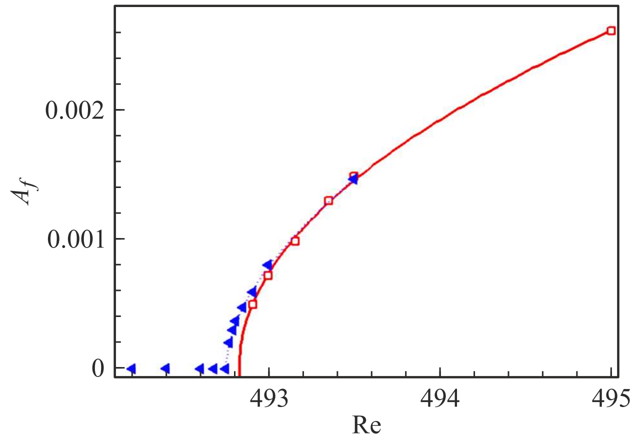

Figure 4. Amplitude of oscillations of the azimuth velocity A_f [m/s] in dependence on the Re number at $\mathrm{R} = 0.67$ and $\theta = 52.9°$. Open symbols (red) — $N = 0$, close symbols (blue) — $N = 0.04$. The solid (red) line — the parabolic approximation of the dependence of A_f on the Re number at $N = 0$. The zero values of A_f correspond to the calculations, which had evident time-continuous decrease in the value A_f in more than 1000 times in comparison with the initial value.

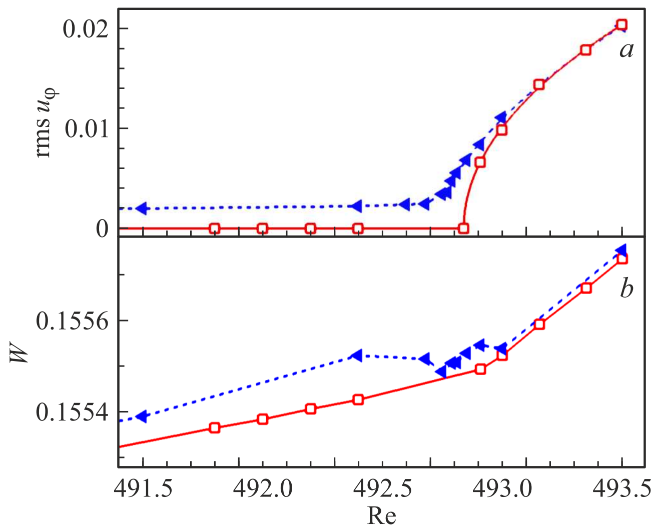

Figure 5. Dependence of the mean-square deviations of the azimuth component of the flow velocity (*a*) and the normalized averaged flow velocity (*b*) on the Re number. Open symbols (red) — $N = 0$, close symbols (blue) — $N = 0.04$.

Comparison of the results of determination of the stability limit position as calculated by the different methods

Kind of solution	Re_{c0} $(N = 0)$	Re_c $(N = 0.04)$	$(Re_{c0} - Re_c)/Re_{c0}$
Analysis the linear stability (from [4])	489.284	489.246	$7.7665 \cdot 10^{-5}$
Non-linear solution, the known method	492.822	492.761 ± 0.011	$1.2378 \cdot 10^{-4}$
Non-linear solution, the proposed method	492.822	492.765 ± 0.001	$1.1566 \cdot 10^{-4}$

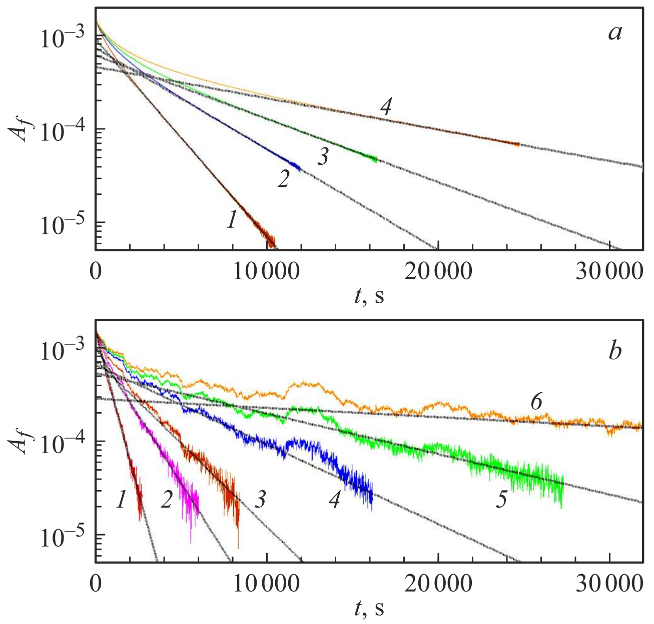

Figure 6. Dependence of A_f [m/s] on the time t (the solid lines) and the exponential approximation of this dependence (the dashed lines) without noise (a) and with noise (b). a — Re = 492.4 (*1*), 492.6 (*2*), 492.68 (*3*), 492.75 (*4*); b — Re = 491.5 (*1*), 492.2 (*2*), 492.4 (*3*), 492.6 (*4*), 492.68 (*5*), 492.75 (*6*).

noise, as mentioned above, the stability limit position is assumed to be determined by zero tending of the parabolic approximation $A_f \sim (\mathrm{Re} - \mathrm{Re}_c)^{1/2}$. With the final noise amplitude, these approximations are unknown, and in this case the accuracy of determination of the stability limit can be increased by refining the dependence of A_f on the Re number at $(\mathrm{Re} - \mathrm{Re}_c) \to 0$. However, when approaching the stability limit, the transient time between the initial and final values A_f can be significantly increased. Thus, at Re = 492.77 the required calculation time is increased to $4.8 \cdot 10^4$ s, which is more than $3.3 \cdot 10^4$ revolutions of the inner sphere, which in 1.5 times more than for Re = 492.9. At the same time, it is unknown in advance whether it is possible to get the time-independent averaged value A_f, because, as noted above, when $\mathrm{Re} < \mathrm{Re}_c$ A_f is constantly decreasing. In this regard, it is interesting to consider other approaches to determining the dependence of A_f on the Re number.

In the presence of noise, one of these approaches is based on the properties of the amplitudes of the azimuth modes of the flow. When $\mathrm{Re} < \mathrm{Re}_c$, the amplitudes of the azimuth modes exponentially decay in time, including in the presence of noise [4]. Let us consider how the values A_f change, when $\mathrm{Re} < \mathrm{Re}_c$ (Fig. 6). It is clear that both at the stationary boundary conditions (Fig. 6, a) and at random fluctuations of the rotational rate (Fig. 6, b), after fast decrease of the initial interval there is the exponential dependence of A_f on t: $A_f \sim \exp(-\mathrm{B}t)$, where B — the decay decrement. With increase in the Re number, the decay decrement B is decreasing and the dependence A_f on t is tending to the horizontal line, as the condition B = 0 is also satisfied when $\mathrm{Re} = \mathrm{Re}_c$. Figure 7 shows the dependence of the decay decrements on the Re number, which allows determining the values Re_c both without noise and in the presence of noise from the condition B = 0. It is clear that near the stability limit the dependence of B on the R number is close to the linear one, while the value Re_c in the presence of noise is less than the similar value at the stationary boundary conditions, which corresponds to the results obtained with the traditional approach (Fig. 4). Re_c obtained by the different methods are compared in the table to be indicative of good compliance of the results.

The results shown on Figs. 7 and 6 demonstrate the advantages of the proposed approach to determine the critical number Re_c both without noise and in the presence of noise. First of all, the near-linear kind of the dependence B(Re) when $\mathrm{Re} < \mathrm{Re}_c$ allows reducing the required number of the calculations at the various Re numbers, thereby naturally resulting in reduction of a calculation scope. Secondly, the exponential time dependence A_f allows reducing the

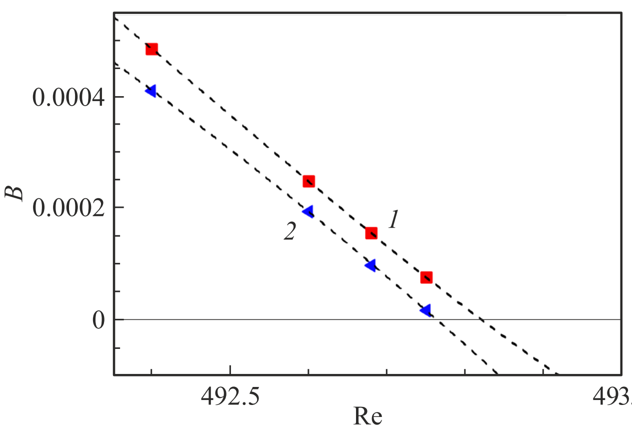

Figure 7. Dependence of the decay decrements B on the Re number. *1* — (the red symbols) — $N = 0$, *2* — (the blue symbols) — $N = 0.04$. The dashed lines — data approximation for *1* and *2*, the horizontal line — $B = 0$.

duration of the time sampling at each value Re, thereby resulting in reduction of the calculation efforts.

Conclusion

The present study has numerically investigated the impact of the random fluctuations added to the constant averaged rotational rate of the inner sphere on the flow stability of the viscous incompressible fluid in the rotational spherical layer. It has shown an analogy between the noise impact and the impact of the periodic oscillations on the distribution of the rate fluctuations in the flow prior to the stability loss. As in the periodic modulation of the rotational rate of the internal sphere, the averaged flow rates grow under noise impact. It is found that the increase in the averaged flow velocities by the impact of noise is higher in the flows prior to the stability loss in comparison with the flows after the stability loss.

It has been demonstrated that in comparison with the stationary boundary conditions, adding the noise reduces the critical value of the Reynolds number Re_c corresponding to the flow stability limit. This result has been obtained by the traditional method from the analysis of the dependence of the amplitude of the flow velocity oscillations on the Reynolds numbers when $\mathrm{Re} > \mathrm{Re}_c$. It has proposed the new method of determining the value Re_c based on the analysis of the amplitude behavior of the flow velocity oscillations at the Reynolds numbers less than Re_c, i.e. the exponential decay of the oscillation amplitude in time and the near-linear dependence of the decay decrement on the Re number. The results obtained by the different methods have shown good compliance. It has demonstrated the reducibility of the calculation scope when applying the proposed approach. It is still unclear whether it is possible to apply this approach not only when adding the noise to the constant rotational rate, but at another behavior of the time dependence of the boundary conditions, for example, in the periodic oscillations of the rotational rate, which will be investigated in the future studies.

Conflict of interest

The authors declare that they have no conflict of interest.

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