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Transverse components of the electromagnetic field in a waveguide with modulated in space and in time magnetodielectric filling

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Received June 12, 2022
Revised July 15, 2022
Accepted July 15, 2022

The propagation of transverse magnetic (TM) and transverse electric (TE) electromagnetic waves in a regular ideal waveguide of arbitrary cross section is considered. It is assumed that the permittivity and permeability of the magnetodielectric filling of the waveguide are functions that depend on the coordinate and time. Analytical expressions for the transverse components of the magnetic and electric vectors of the TM- and TE-fields in the waveguide are obtained from the system of Maxwell equations. They are expressed in terms of the longitudinal components of the electric and magnetic vectors, which describe the transverse magnetic and transverse electric fields in the waveguide. For the above longitudinal components of the electric and magnetic vectors, the wave equations are given, which are also obtained from the system of Maxwell's equations.

Keywords: Maxwell equations, propagation of electromagnetic waves, waveguide with modulated filling, transverse components, Helmholtz equations, Dirichlet and Neumann problems.

DOI: 10.21883/EOS.2022.10.54865.3813-22

Introduction

The study of the features of propagation for electromagnetic waves in unlimited and limited media (in particular, in waveguides of arbitrary cross section), the dielectric and magnetic permeability of which depend on the coordinate and time (especially when this dependence is periodic), is one of the main problems of electromagnetic theory. Such studies in waveguides are of great importance both for the development of the theory of electrodynamics of modulated media, and for the possibilities of practical application of the above waveguides in microwave electronics [1–7]. In the present work, it is shown that the longitudinal components of the electric and magnetic vectors $(E_z(x, y, z, t), H_z(x, y, z, t))$, which satisfy the corresponding wave equations can determine the TM- and TE-fields in a waveguide with modulated filling. Analytical expressions for the transverse components of the electric and magnetic vectors are obtained from the system of Maxwell equations, expressed in terms of the longitudinal components for the TM- and TE-fields in a waveguide with a modulated magnetodielectric filling. Using these analytical expressions, according to the Poynting theorem, one can calculate the energies of the transition radiation of a charged particle during its uniform motion in a waveguide (along the axis or perpendicular to the axis), the magnetodielectric filling of which is modulated in space and time, in particular, according to the periodic law.

Problem statement and its solution

Let an electromagnetic wave with frequency ω_0 propagate along the axis of an ideal regular waveguide of arbitrary cross section. It is assumed that the wave propagates in the positive direction of the OZ axis of some Cartesian coordinate system (the OZ axis coincides with the waveguide axis), and the permittivity and permeability of filling the waveguide with the pump wave are modulated in space and time $(\varepsilon(z, t), \mu(z, t))$, for example, according to the harmonic law under the condition of small modulation depths (in practice they are by the order of 10^{-4}, see, for example, formulas (1) and (2) from [7]). Note that the transverse magnetic field in the waveguide ($H_z = 0$, $E_z \neq 0$) in the problem under consideration is completely described using the longitudinal component of the electric vector $E_z(x, y, z, t)$ [2]. From Maxwell's system of equations

$$\text{I. rot}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad \text{II. rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\text{III. div}\mathbf{D} = 0, \quad \text{IV. div}\mathbf{B} = 0, \tag{2}$$

$$\mathbf{D} = \varepsilon_0 \varepsilon(z, t)\mathbf{E}, \quad \mathbf{B} = \mu_0 \mu(z, t)\mathbf{H},$$

where $\varepsilon_0 = (36\pi \cdot 10^9)^{-1}$ F/m is electrical constant, $\mu_0 = 4\pi \cdot 10^{-7}$ H/m is magnetic constant, for the longitudinal component of the electric vector $(E_z(x, y, z, t))$ one can get the wave equation of the form

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{1}{\varepsilon}\frac{\partial(\varepsilon E_z)}{\partial z}\right) - \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(\mu \frac{\partial(\varepsilon E_z)}{\partial t}\right) = 0. \tag{3}$$

In the particular case of periodic dependence of ε and μ on z and on t and assuming small modulation depths, equation (3) can be solved by the method developed by the author in [2], and find an analytic expression for $E_z(x, y, z, t)$.

The system of Maxwell's equations allows one to express the transverse components of the electric and magnetic vectors of the TM-field in terms of $E_z(x, y, z, t)$. In fact, it follows from the second Maxwell equation that

$$(\mathrm{rot}\mathbf{E})_z = -\mu_0 \frac{\partial(\mu H_z)}{\partial t}. \qquad (4)$$

And since for the TM-field $H_z = 0$, then $(\mathrm{rot}\mathbf{E})_z = 0$ or

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \qquad (5)$$

Note that from the third Maxwell equation we have

$$\mathrm{div}\mathbf{D} = \mathrm{div}(\varepsilon_0 \varepsilon \mathbf{E}) = \varepsilon_0 \frac{\partial(\varepsilon E_x)}{\partial x} + \varepsilon_0 \frac{\partial(\varepsilon E_y)}{\partial y} + \varepsilon_0 \frac{\partial(\varepsilon E_z)}{\partial z} = 0$$

or

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{1}{\varepsilon} \frac{\partial(\varepsilon E_z)}{\partial z}. \qquad (6)$$

Differentiating (5) with respect to y and (6) with respect to x, we obtain the following system of equations:

$$\begin{cases} \dfrac{\partial^2 E_y}{\partial x \partial y} - \dfrac{\partial^2 E_x}{\partial y^2} = 0, \\ \dfrac{\partial^2 E_x}{\partial x^2} + \dfrac{\partial^2 E_y}{\partial y \partial x} = -\dfrac{1}{\varepsilon} \dfrac{\partial}{\partial x}\left(\dfrac{\partial(\varepsilon E_z)}{\partial z}\right). \end{cases} \qquad (7)$$

From system (7), taking into account the fact that

$$\frac{\partial^2 E_y}{\partial y \partial x} = \frac{\partial^2 E_y}{\partial x \partial y},$$

(second-order mixed partial derivatives are equal, since they are continuous functions in x and y), it follows

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} = -\frac{1}{\varepsilon} \frac{\partial}{\partial x}\left(\frac{\partial(\varepsilon E_z)}{\partial z}\right). \qquad (8)$$

Note that similarly from (5) and (6) we have

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = -\frac{1}{\varepsilon} \frac{\partial}{\partial y}\left(\frac{\partial(\varepsilon E_z)}{\partial z}\right). \qquad (9)$$

Further, using the first and fourth Maxwell equations and taking into account that the second-order mixed partial derivatives of H_x and H_y with respect to the variables x and y are equal, we can get

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} = -\varepsilon_0 \frac{\partial}{\partial y}\left(\frac{\partial(\varepsilon E_z)}{\partial t}\right), \qquad (10)$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} = \varepsilon_0 \frac{\partial}{\partial x}\left(\frac{\partial(\varepsilon E_z)}{\partial t}\right). \qquad (11)$$

When solving problems of electrodynamics in waveguides filled with a magnetodielectric medium, the permittivity and magnetic permeability of which depend on z and on t, in the case of a TM field, the vectors of the electric and magnetic fields are represented as expansions in proper functions $\psi_n(x, y)$ of the first boundary value problem (Dirichlet problem) for the cross section of the waveguide, i.e.

$$\begin{aligned} \mathbf{E}(x, y, z, t) &= \sum_{n=0}^{\infty} \mathbf{E}_n(z, t)\psi_n(x, y), \\ \mathbf{H}(x, y, z, t) &= \sum_{n=0}^{\infty} \mathbf{H}_n(z, t)\psi_n(x, y). \end{aligned} \qquad (12)$$

In (12) the orthonormal functions $\psi_n(x, y)$ satisfy the Helmholtz equation with the corresponding boundary condition [6]:

$$\begin{gathered} \frac{\partial^2 \psi_n(x, y)}{\partial x^2} + \frac{\partial^2 \psi_n(x, y)}{\partial y^2} + \lambda_n^2 \psi(x, y) = 0, \\ \psi_n(x, y)\Big|_{\Sigma} = 0, \end{gathered} \qquad (13)$$

where λ_n is eigenvalues of the Dirichlet problem, Σ is the contour of the waveguide cross section.

According to (12), the components of the electric and magnetic vectors of the TM-field have the form

$$\begin{aligned} E_z(x, y, z, t) &= \sum_{n=0}^{\infty} E_n^z(z, t)\psi_n(x, y) = \\ &= \sum_{n=0}^{\infty} E_n(z, t)\psi_n(x, y), \end{aligned} \qquad (14)$$

$$E_x(x, y, z, t) = \sum_{n=0}^{\infty} E_n^x(z, t)\psi_n(x, y), \qquad (15)$$

$$E_y(x, y, z, t) = \sum_{n=0}^{\infty} E_n^y(z, t)\psi_n(x, y), \qquad (16)$$

$$H_x(x, y, z, t) = \sum_{n=0}^{\infty} H_n^x(z, t)\psi_n(x, y), \qquad (17)$$

$$H_y(x, y, z, t) = \sum_{n=0}^{\infty} H_n^y(z, t)\psi_n(x, y). \qquad (18)$$

Now, using (8)–(11), (13) and (14)–(18) for the transverse components of the electric and magnetic vectors, we obtain the following expressions:

$$E_x(x, y, z, t) = = \frac{1}{\varepsilon(z, t)} \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial(\varepsilon(z, t)E_n(z, t))}{\partial z} \frac{\partial \psi_n(x, y)}{\partial x},$$

$$E_y(x,y,z,t) = = \frac{1}{\varepsilon(z,t)} \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial(\varepsilon(z,t)E_n(z,t))}{\partial z} \frac{\partial \psi_n(x,y)}{\partial y},$$

$$H_x(x,y,z,t) = = \varepsilon_0 \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial(\varepsilon(z,t)E_n(z,t))}{\partial t} \frac{\partial \psi_n(x,y)}{\partial y},$$

$$H_y(x,y,z,t) = = -\varepsilon_0 \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial(\varepsilon(z,t)E_n(z,t))}{\partial t} \frac{\partial \psi_n(x,y)}{\partial x}$$

or

$$\mathbf{E}_\tau^{(TM)}(x,y,z,t) = = \frac{1}{\varepsilon(z,t)} \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial(\varepsilon(z,t)E_n(z,t))}{\partial z} \nabla\psi_n(x,y), \quad (19)$$

$$\mathbf{H}_\tau^{(TM)}(x,y,z,t) = = -\varepsilon_0 \sum_{n=0}^{\infty} \lambda_n^{-2} \frac{\partial(\varepsilon(z,t)E_n(z,t))}{\partial t} [\mathbf{z}_0 \nabla\psi_n(x,y)], \quad (20)$$

where $\nabla = \mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y)$, $\mathbf{z}_0 = \{0, 0, 1\}$, index τ indicates transverse components.

Also note that if the problems of electrodynamics in waveguides relate to the transverse electric field ($E_z = 0, H \neq 0$), then we use the expansion of the vectors $\mathbf{E}(x,y,z,t)$ and $\mathbf{H}(x,y,z,t)$ with respect to the proper functions $\widehat{\psi}_n(x,y)$ of the second boundary value problem for the waveguide cross section (the Neumann problem), i.e.,

$$\mathbf{E}(x,y,z,t) = \sum_{n=0}^{\infty} \mathbf{E}_n(z,t)\widehat{\psi}_n(x,y),$$
$$\mathbf{H}(x,y,z,t) = \sum_{n=0}^{\infty} \mathbf{H}_n(z,t)\widehat{\psi}_n(x,y), \quad (21)$$

where the functions $\widehat{\psi}_n(x,y)$ satisfy the Helmholtz equation with the corresponding boundary condition [6]:

$$\frac{\partial^2 \widehat{\psi}_n(x,y)}{\partial x^2} + \frac{\partial^2 \widehat{\psi}_n(x,y)}{\partial y^2} + \widehat{\lambda}_n^2 \widehat{\psi}_n(x,y) = 0,$$
$$\left.\frac{\partial^2 \widehat{\psi}_n(x,y)}{\partial \mathbf{n}}\right|_\Sigma = 0. \quad (22)$$

In (22) $\widehat{\lambda}_n^2$ are eigenvalues of the Neumann problem, and $\mathbf{n}$ is normal vector k Σ. In the case of a TE-field, it is similarly possible to express the transverse components of the electric and magnetic vectors from the system of Maxwell equations in terms of the longitudinal component of the magnetic vector, and also to obtain the wave equation for $H_z(x,y,z,t)$. The calculations lead to the following expressions:

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{1}{\mu}\frac{\partial(\mu H_z)}{\partial z}\right) - - \varepsilon_0\mu_0 \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial(\mu H_z)}{\partial t}\right) = 0, \quad (23)$$

$$\mathbf{E}_\tau^{(TE)}(x,y,z,t) = = \mu_0 \sum_{n=0}^{\infty} \widehat{\lambda}_n^{-2} \frac{\partial(\mu(z,t)H_n(z,t))}{\partial t} [\mathbf{z}_0 \nabla\widehat{\psi}_n(x,y)], \quad (24)$$

$$\mathbf{H}_\tau^{(TE)}(x,y,z,t) = = \frac{1}{\mu(z,t)} \sum_{n=0}^{\infty} \widehat{\lambda}_n^{-2} \frac{\partial(\mu(z,t)H_n(z,t))}{\partial z} \nabla\widehat{\psi}_n(x,y),$$

Conclusion

The results of this work show that the method of expanding the electric and magnetic vectors of TM and TE-fields in a waveguide in terms of proper functions of the first and second boundary value problems for the cross section of the waveguide (the Dirichlet and Neumann problems) when solving the problems of propagation of electromagnetic TM- and TE-waves in A waveguide with a nonstationary and inhomogeneous magnetodielectric filling allows the transverse components of the TM and TE fields to be expressed in terms of the longitudinal components of the electric and magnetic vectors. The analytical expressions obtained for the transverse components of the TM- and TE-fields are of great importance in solving problems of transition and Cherenkov radiation from uniformly moving sources in a waveguide with a spatially and temporally modulated filling. Using them, according to the Poynting theorem, it is possible to calculate the radiation energy.

Conflict of interest

The author declares that he has no conflict of interest.

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