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Different modes of three coupled generators capable of demonstrating quasiperiodic oscillations

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The dynamics of three coupled generators capable of demonstrating autonomous quasiperiodic oscillations is considered. The complex structure of Lyapunov charts of the system revealing invariant tori of different (high) dimensions, quasiperiodic bifurcations, Arnold resonance web, and other features is discussed. There was revealed the possibility of four−frequency tori in case of individual subsystems that exhibit the limit cycle mode.

Keywords: generator, quasi-periodic oscillations, invariant tori, Lyapunov exponents.

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Quasi-periodic oscillations are an independent class quite common in science and technology [1]. The best known and most comprehensively studied oscillations are those arising in interactions between self-oscillating subsystems with periodic regimes. On the other hand, paper [2] considers a case of an electronic system with an autonomous quasiperiodic dynamics. In addition to [2], relatively recently a few versions of similar systems (quasiperiodic generators) were proposed [3–5], which were implemented experimentally. Taking them into account made it possible to define a wide range of tasks. For instance, the problems of dynamics of autonomous generators, generators under external forces, two coupled generators, and external excitation of two coupled generators were considered sequentially [3–9]. Notice that the problem of quasiperiodic bifurcations (bifurcations of invariant tori) is also associated with this subject [10–14]. As known from the oscillation theory, an increase in the number of interacting subsystems essentially enriches the dynamics. For the purpose of developing the problem of dynamics of two quasiperiodic generators [7], let us consider the case of three coupled subsystems. Let us choose the chain-type coupling (the case of the ring-type coupling is a separate task). As a tool, we will use the Lyapunov analysis as in [7]. This analysis is capable of revealing the global structure of the parameter plane, including such aspects as constructing the hierarchy of different−dimensional tori, and very complex structures like the Arnold resonance web [15], which seems yet unattainable by other methods. Notice that the Lyapunov analysis allows revealing local torus bifurcations by the method presented in [10] with the accuracy sufficient for physical problems.

Let us write the set of equations in a manner similar to that for two generators [7]:

$$\ddot{x}_1 - (\lambda_1 + z_1 + x_1^2 - \beta x_1^4)\dot{x}_1 + \omega_0^2 x_1 + M_C(\dot{x}_1 - \dot{x}_2) = 0,$$

$$\dot{z}_1 = b(\varepsilon - z_1) - k\dot{x}_1^2,$$

$$\ddot{x}_2 - (\lambda_2 + z_2 + x_2^2 - \beta x_2^4)\dot{x}_2 + (\omega_0 + \Delta_1)^2 x_2 + M_C(2\dot{x}_2 - \dot{x}_1 - \dot{x}_3) = 0,$$

$$\dot{z}_2 = b(\varepsilon - z_2) - k\dot{x}_2^2,$$

$$\ddot{x}_3 - (\lambda_3 + z_3 + x_3^2 - \beta x_3^4)\dot{x}_3 + (\omega_0 + \Delta_2)^2 x_3 + M_C(\dot{x}_3 - \dot{x}_2) = 0,$$

$$\dot{z}_3 = b(\varepsilon - z_3) - k\dot{x}_3^2. \quad (1)$$

Here x, z are the generator variables, ω_0 is the first generator eigen frequency, $\Delta_{1,2}$ are the frequency mismatches of the second and third generators with respect to the first one, M_C is the coupling coefficient. As per [6,7], let us choose the following parameters: $\varepsilon = 4$, $b = 1$, $k = 0.02$, $\beta = 1/18$, $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

In an individual generator, two-frequency quasiperiodic modes T_2 are observed in a certain variation range of parameter ω_0 except for very narrow resonance intervals. Fig. 1 presents a relevant illustration in the form of a bifurcation diagram constructed for the Poincare, section with plane $\dot{x} = 0$, with which the set of points $(x_s, 0, z_s)$ is

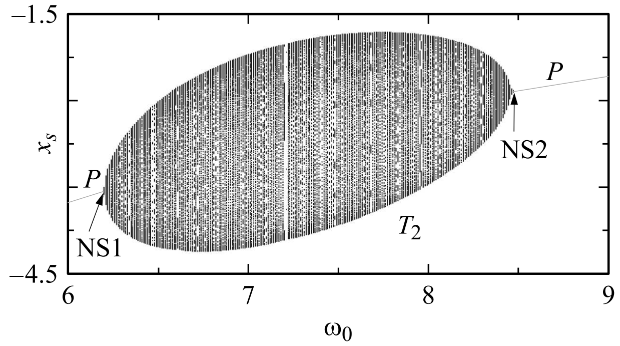

Figure 1. Bifurcation diagram of an individual generator of quasiperiodic oscillations. NS1 and NS2 are the Neimark−Sacker bifurcation points.

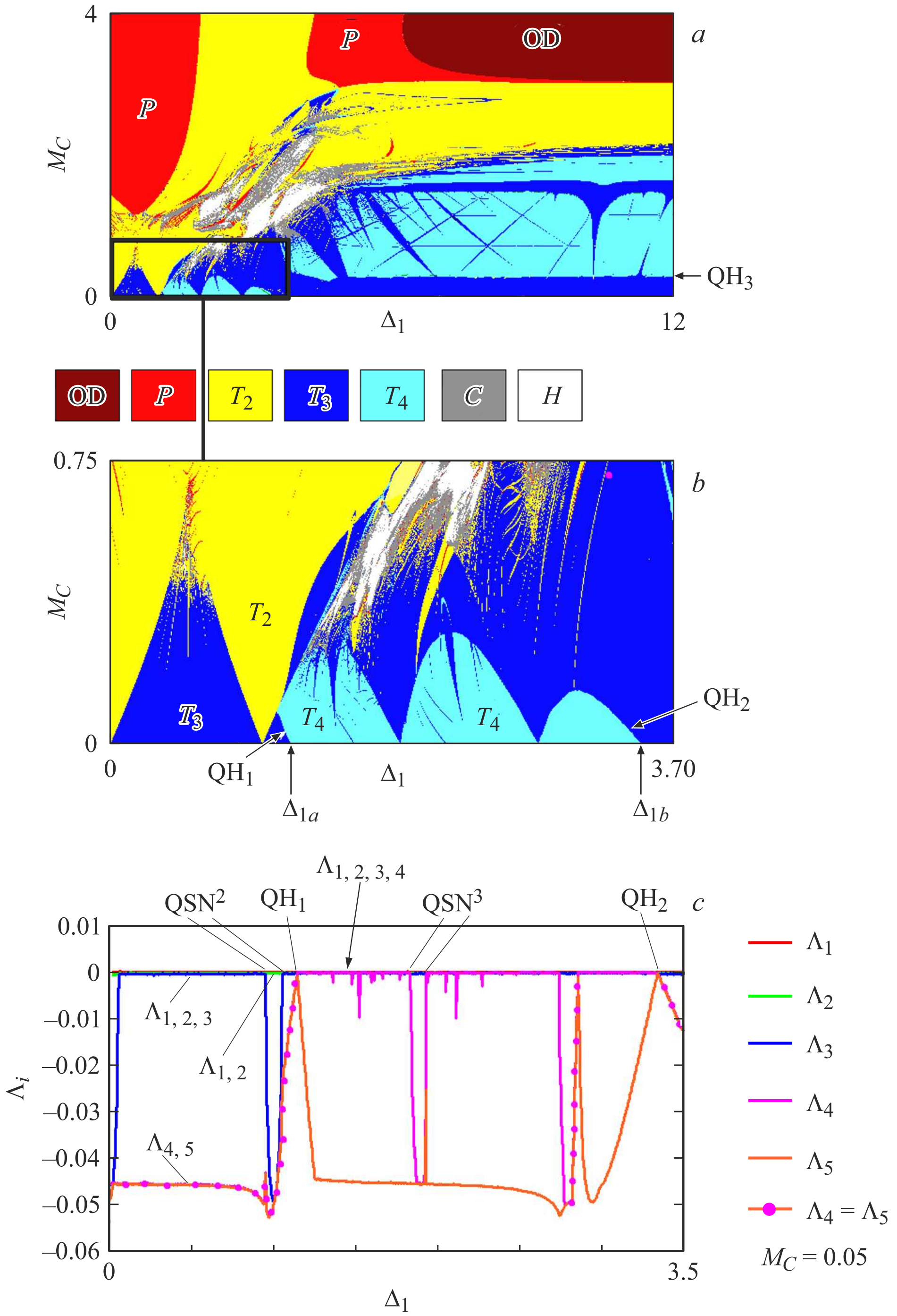

Figure 2. The chart of Lyapunov exponents (*a*), its fragment (*b*), plots of Lyapunov exponents curves (*c*). QSN^2 are the saddle–node bifurcations of two-frequency tori, QSN^3 are the saddle–node bifurcations of three-frequency tori, QH are the quasiperiodic Hopf bifurcations of three-frequency tori. The colored figure is available in the electronic version of the paper.

associated. The diagram presented in Fig. 1 is given in the projection on variable x_s. At the range limits NS1 ($\omega_0 \approx 6.201$) and NS2 ($\omega_0 \approx 8.45$), the Neimark–Sacker bifurcations get realized, while beyond the range the limit cycle mode P is observed.

For the system of coupled generator (1), let us first select frequency parameters $\omega_0 = 5$ and $\Delta_2 = 1$ at which the first and third generators exhibit the mode of periodic oscillations. In this case, variations in parameter Δ_1 exhibit that the second generator switches from the limit cycle mode to the quasiperiodic mode at $\Delta_{1a} = 1.201$ and back at $\Delta_{1b} = 3.45$. Fig. 2 demonstrates the Lyapunov chart of set (1) and its zoomed fragment on the plane (Δ_1, M_C). We can see areas of four— (T_4), three— (T_3) and two-

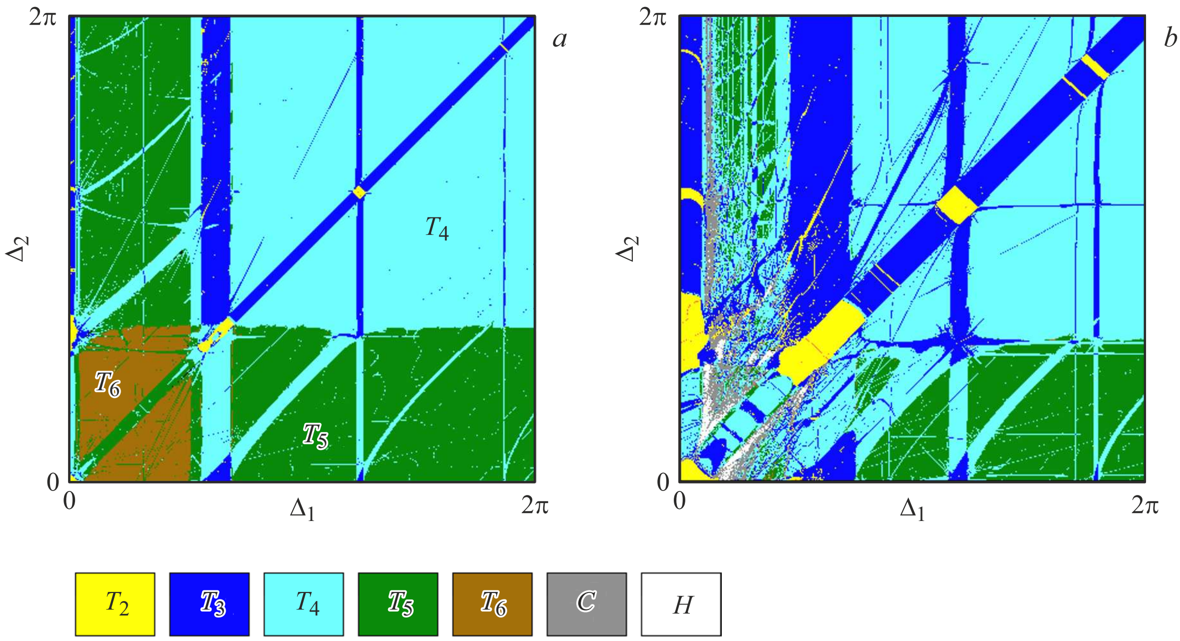

Figure 3. Lyapunov exponent charts on the plane of the generators frequency mismatches. $M_C = 0.1$ (a) and 0.3 (b). The colored figure is available in the electronic version of the paper.

frequency (T_2) tori, periodic modes (P), oscillation death (OD), chaos (C) and hyperchaos (H). The mode type may be determined from the spectrum of Lyapunov exponents as in [7]; dimensions of invariant tori are defined by the number of zero exponents. Let us discuss the chart structure more comprehensively. Area $\Delta_1 < \Delta_{1a}$ is organized simpler than others, namely, here all the individual generators are in the limit cycle mode. In this case, the zoomed fragment of the chart (Fig. 2, b) demonstrates two tongues of two-frequency tori T_2 between which there is the area of three-frequency quasiperiodicity T_3. At a large coupling (Fig. 2, a), overlap of the tongues gives rise to the periodic mode P of the complete synchronization of three generators. In this case, the pattern is similar to that for three coupled van der Pol generators [16]. Differences emerge when frequency parameter Δ_1 passes through value Δ_{1a} at which the second generator switches to the mode of quasiperiodic oscillations. Accordingly, the area of four-frequency quasiperiodicity T_4 appears in the chart (Fig. 2, b). This transition proceeds through quasiperiodic Hopf bifurcation QH_1. In Fig. 2, b, the line of this bifurcation originates from point $\Delta_1 = \Delta_{1a}$. This bifurcation may be diagnosed using the Lyapunov exponent curves by the method given in [10]. The condition for its occurrence is the equality of two largest non-zero exponents $\Lambda_4 = \Lambda_5 < 0$ before the bifurcation threshold. At the moment of bifurcation, exponent Λ_4 becomes non-zero, while Λ_5 becomes negative again. Thereat, torus T_4 softly arises from torus T_3. In its turn, the appeared area of four-frequency tori contains a submerged set of tongues of three-frequency tori having tips on the frequency mismatch axis Δ_1. Fig. 2, c demonstrates the boundaries of one of those tongues QSN^3 associated with saddle−node bifurcations of three-frequency tori. The bifurcation of this type is characterized by the fact that, when the parameter is varied, only one Lyapunov exponent approaches zero and gets zero [10], namely, Λ_4, while $\Lambda_5 < 0$ (Fig. 2, c). Then, when the tongue boundary is crossed, stable torus T_3 collides with the saddle−type torus of the same dimension, which gives rise to torus T_4 [10]. Notice also that chaos C and hyperchaos H areas appear with increasing coupling strength when the tori get damaged. In passing through $\Delta_1 = \Delta_{1b}$, the second generator again switches to the limit cycle mode, and three-frequency tori again arise through the quasiperiodic Hopf bifurcation QH_2 that is passed-through in the reverse order (Fig. 2, b, c). Let us return to Fig. 2, a. One can see that at high Δ_1 values, as the coupling strength increases, three-frequency tori transform back to four-frequency ones. This transition is also associated with the quasiperiodic Hopf bifurcation QH_3. Such tori emerge despite all three individual generators are in the limit cycle mode, which is one more fact distinguishing this case from the case of three van der Pol oscillators [16]. This area of four-frequency tori is pierced by narrow strips of three-frequency tori and forms a structure named as the Arnold resonance web [15]. As the coupling increases, there become possible two-frequency tori corresponding to a broad strip free of resonance inclusions. Then an extremely narrow periodic-mode area arises, while at $M_C > 3.0$ only the equilibrium state remains stable: the oscillation death (OD) mode is observed.

We have discussed the case when only one individual generator exhibits quasiperiodic oscillations. Now let us vary both the frequency mismatches (Δ_1 and Δ_2). Assume that parameter ω_0 is equal to 2π, which is relevant to the quasiperiodic–oscillations mode in the first autonomous generator. Therefore, all the three generators can exhibit quasiperiodic oscillations in the autonomous mode. To visualize the emerging modes, in this case it is convenient to use the Lyapunov exponent chart on the plane of frequency mismatches (Δ_1,Δ_2) shown in Fig. 3 at two fixed coupling strengths. It is clearly seen that now six-frequency T_6 and five-frequency T_5 tori are possible at a small coupling. With increasing coupling, six-frequency tori fully disappear, while areas of the five-frequency ones decrease significantly. A set of characteristic intersecting strips in the form of the Arnold resonance web is also observed. Now such a structure arises based not only on four-frequency tori but also on five-frequency ones. Notice that areas of the two-frequency tori are quite small and correspond to the intersection of resonance strips of the three-frequency tori. Periodic modes (the complete-synchronization modes) are absent at these coupling levels.

Hence, the problems on quasiperiodic generator dynamics appear to be rather complex. A logical development of the case of two coupled generators is switching to the analysis of three coupled generators. Such a system may be studied, to a large extent, by using the two-parameter Lyapunov analysis. In the case of quasiperiodic oscillations in one individual generator, the pattern proves to be different from that in the case of three coupled van der Pol generators. The quasiperiodic Hopf bifurcations, saddle–node bifurcations of invariant tori and Arnold resonance web are observed. Four-frequency tori emerge also in case three individual generators are in the limit cycle mode, which is one more fact distinguishing this case from the case of van der Pol oscillators. In the case of quasiperiodic oscillations, each individual generator exhibits emergence of high-dimensional tori and Arnold resonance web based on different-dimensional tori.

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Conflict of interests

The authors declare that they have no conflict of interests.

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