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High-frequency asymptotics of one integral in the theory of equilibrium radiation of electron gas

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It is considered an improper integral that determines the high-frequency asymptotics of the spectral energy distribution of equilibrium radiation in an ideal electron gas. It has been established that in the „high-frequency" limit the asymptotics of this integral has a power-law character, and its value is proportional to the density of the electron gas as a function of temperature and chemical potential for arbitrary degeneracy of electrons.

Keywords: equilibrium radiation, spectral energy distribution, electron gas.

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In recent years, there has been an upsurge in interest in warm dense matter (WDM) studies (see [1–4] for details). This is attributable to the accumulation of experimental data in the field of laboratory astrophysics, where the influence of matter on the characteristics of equilibrium radiation in WDM is a vital point [5,6]. The thing is that the spectral energy distribution of equilibrium radiation established by Planck corresponds to the idealized model of a blackbody in a void filled with radiation and a confined absorbing material medium, and radiation is thus in thermodynamic equilibrium with the surrounding matter (see [7] for details). The effects of interaction of photons with matter at the boundary of the void are often neglected, although this interaction is the one that establishes the blackbody equilibrium [7]. In addition, the available experimental data [8,9] suggest that matter affects the characteristics of intrinsic radiation of a uniform and isotropic material medium. Various aspects of determination of the characteristics of equilibrium electromagnetic radiation in the presence of matter were discussed in [10–13] and papers cited there. Specifically, it was found in [10] that mean energy E_{ph} of equilibrium radiation in a material medium occupying macroscopic volume V may be presented as

$$E_{ph}=V\sum_{\lambda=1}^{2}\int d^3q\hbar\omega_{\mathbf{q}}f(\mathbf{q},\lambda)/(2\pi)^3=V\int_0^\infty d\omega\varepsilon_\omega(T,\{\mu_a\}):$$

$$\varepsilon_\omega(T,\{\mu_a\})=\varepsilon_\omega^{(0)}(T)+\Delta\varepsilon_\omega(T,\{\mu_a\}),$$

$$\varepsilon_\omega^{(0)}=\frac{\hbar}{\pi^2c^3}\frac{\omega^3}{\exp(\hbar\omega/T)-1}, \qquad (1)$$

$$\Delta\varepsilon_\omega(T,\{\mu_a\})=\frac{\hbar\omega^3}{\pi^2c^3}\,\mathrm{cth}\left(\frac{\hbar\omega}{T}\right)$$

$$\times\left(\frac{c^5}{\pi\omega}\int_0^\infty dq\,q^4\frac{\mathrm{Im}\varepsilon_T^{\mathrm{tr}}(q,\omega)}{|\varepsilon_T^{\mathrm{tr}}(q,\omega)\omega^2-c^2q^2|^2}-\frac{1}{2}\right). \qquad (2)$$

Here and elsewhere, $f(\mathbf{q},\lambda)$ is the exact equilibrium distribution function of photons over momenta $\hbar\mathbf{q}$ and polarization $\lambda=1,2$ for the studied system that is characterized by thermodynamic temperature T (in energy units). An equilibrium uniform and isotropic material medium is regarded as an aggregate of an intrinsic quantized electromagnetic field and charged particles. This implies that spectral energy distribution of radiation in matter $\varepsilon_\omega(T,\{\mu_a\})$ depends not only on frequency ω and temperature T (as in Planck formula $\varepsilon_\omega^{(0)}(T)$ (1) for an ideal photon gas), but also on the characteristics of matter: the set of chemical potentials $\{\mu_a\}$ of charged particles of various types a found in the system. The result is that the difference between the spectral energy distribution of radiation in matter and Planck formula $\varepsilon_\omega^{(0)}(T)$ (1) is defined completely by transverse permittivity $\varepsilon_T^{\mathrm{tr}}(q,\omega)$ for the considered uniform and isotropic medium with its linear electromagnetic properties specified unambiguously by longitudinal $\varepsilon_T^{l}(q,\omega)$ and transverse $\varepsilon_T^{\mathrm{tr}}(q,\omega)$ permittivities [14]. Lower index T in this notation indicates that the corresponding functions are considered in the thermodynamic limit $(V\to\infty):\varepsilon_T^{l(\mathrm{tr})}(q,\omega)\equiv\varepsilon^{l(\mathrm{tr})}(q,\omega;T,\{\mu_a\})$.

Further examination involves analyzing the convergence of the improper integral at the right-hand part of relation (2) [11,13] with account for the general properties of electrodynamic response functions [15]. However, in contrast to longitudinal permittivity $\varepsilon_T^{l}(q,\omega)$ (see [16] and references therein), transverse permittivity $\varepsilon_T^{\mathrm{tr}}(q,\omega)$ has remained underexamined. In fact, only an integral representation at arbitrary degeneracy of electrons is available even for an ideal electron gas (see [17] for details).

That said, the authors of [11] have determined the specific features of function $\Delta\varepsilon_\omega(T,\{\mu_a\})$ (2) for an ideal electron gas at arbitrary degeneracy in the low-frequency region $(\omega\to 0)$. It was also demonstrated in [13] that the behavior of function $\Delta\varepsilon_\omega(T,\mu_e)$ (2) for an ideal electron gas at arbitrary degeneracy in the high-frequency limit ($\omega \to \infty$) is characterized by the following expressions:

$$\Delta\varepsilon_\omega(T,\mu_e)|_{\omega\to\infty} \to \frac{2c^2e^2m_e^3}{\pi^3\hbar^3\beta_\omega^4}\Phi(\beta_\omega,\mu_e/T)|_{\omega\to\infty},$$

$$\Phi(\beta,\mu) = \int_0^\infty dx x^2 \ln\left(\frac{1+\exp\big(\mu - x\Delta_x^{(-)}(\beta)\big)}{1+\exp\big(\mu - x\Delta_x^{(+)}(\beta)\big)}\right),$$

$$\beta_\omega = \frac{\hbar\omega}{T}, \quad \Delta_x^{(\pm)}(\beta) = \frac{(\beta/x \pm 1)^2}{4}. \tag{3}$$

Here and elsewhere, m_e is the mass of an electron with charge e and spin $S_e = 1/2$. Thus, the problem reduces to finding the asymptotics of function $\Phi(\beta,\mu)$, which is expressed in terms of improper integral (3), in the region of „high frequencies“ ($\beta \to \infty$) . In order to solve this problem, we present function $\Phi(\beta,\mu)$ in accordance with (3) as $\Phi(\beta,\mu) = \Phi^{(-)}(\beta,\mu) - \Phi^{(+)}(\beta,\mu)$,

$$\Phi^{(\pm)}(\beta,\mu) = 2\beta^3\int_0^\infty dz z^5 \ln\left(1+\exp\left(\mu - \frac{\beta}{4}\left(z \pm \frac{1}{z}\right)^2\right)\right),$$

so that

$$\Delta\varepsilon_\omega(T,\mu_e)|_{\omega\to\infty} \to \frac{2c^2e^2m_e^3}{\pi^3\hbar^3\beta_\omega^4}\Big\{\Phi^{(-)}(\beta_\omega,\mu_e/T)|_{\beta_\omega\to\infty} - \Phi^{(+)}(\beta_\omega,\mu_e/T)|_{\beta_\omega\to\infty}\Big\}. \tag{4}$$

The substitution of variables $y = z \pm 1/z$ yields

$$\Phi^{(+)}(\beta,\mu) = 2\beta^3\int_2^\infty dy\left\{\frac{(z_2^{(+)})^7}{(z_2^{(+)})^2-1} - \frac{(z_1^{(+)})^7}{((z_1^{(+)})^2-1)}\right\} \times \ln\big(1+\exp(\mu - \beta y^2/4)\big), \tag{5}$$

$$\Phi^{(-)}(\beta,\mu) = 2\beta^3\int_{-\infty}^\infty dy \frac{\big(z^{(-)}(y)\big)^7}{\big(z^{(-)}(y)\big)^2+1} \times \ln\big(1+\exp(\mu - \beta y^2/4)\big), \tag{6}$$

$$z_1^{(+)}(y) = \frac{y - \sqrt{y^2-4}}{2}, \quad z_2^{(+)}(y) = \frac{y + \sqrt{y^2-4}}{2},$$

$$z^{(-)}(y) = \frac{y + \sqrt{y^2+4}}{2}. \tag{7}$$

Using relations (5)−(7), we find the „high-frequency“ asymptotics for functions $\Phi^{(\pm)}(\beta,\mu)$ in the $\beta \to \infty$ limit:

$$\Phi^{(+)}(\beta,\mu)|_{\beta\to\infty} \to 4\sqrt{\beta}\int_0^\infty d\alpha\big\{16\beta^{3/2}\alpha + 3\beta^2\big\} \times \exp\big(\mu - \beta - \alpha\sqrt{\beta}\big)|_{\beta\to\infty} \to 12\beta^2\exp(\mu-\beta),$$

$$\Phi^{(-)}(\beta,\mu)|_{\beta\to\infty} \to 4\sqrt{\beta}\int_0^\infty d\alpha(4\beta\alpha^2 + 3\beta^2) \times \ln\big(1+\exp(\mu - \alpha^2/4)\big)|_{\beta\to\infty} \to 12\beta^{5/2}I(\mu),$$

where

$$I(\mu) = \int_0^\infty d\alpha \ln\big(1+\exp(\mu-\alpha^2/4)\big). \tag{8}$$

The high-frequency asymptotics of the spectral energy distribution of equilibrium radiation in an ideal electron gas takes the form

$$\Delta\varepsilon_\omega(T,\mu_e)|_{\omega\to\infty} \to 24c^2e^2m_e^3 I(\mu_e/T)/\pi^3\hbar^3\beta_\omega^{3/2},$$

where function $I(\mu_e/T)$ is defined by relation (8) at arbitrary electron-gas degeneracy $|\mu_e| < \infty$. Then, we take into account that chemical potential of electrons μ_e at a given temperature T is defined by electron number density $n_e(T,\mu_e)$ according to the condition

$$n_e(T,\mu_e) = (2S_e+1)\int d^3p f_e(p)/(2\pi)^3,$$

where

$$f_e(p) = \big\{\exp\big((\epsilon_e(p)-\mu_e)/T\big)+1\big\}^{-1},$$

and $\epsilon_e(p) = \hbar^2p^2/2m_e$ is the energy of a free electron. Integrating (8) by parts, one readily sees that

$$n_e(T,\mu_e) = 2I(\mu_e/T)/\Lambda_e^3\sqrt{\pi},$$

where $\Lambda_e = (2\pi\hbar^2/m_eT)^{1/2}$ is the thermal de Broglie wavelength for electrons. Therefore, $\Phi(\beta_\omega,\mu_e/T)|_{\omega\to\infty} \to 6\sqrt{\pi}n_e(T,\mu_e)\Lambda_e^3(T)\beta_\omega^{5/2}$,

$$\Delta\varepsilon_\omega(T,\mu_e)|_{\omega\to\infty} \to \frac{24\sqrt{2}e^2n_e(T,\mu_e)}{\pi c}\left(\frac{m_ec^2}{\hbar\omega}\right)^{3/2}. \tag{9}$$

Relation (9) agrees with the results obtained for an ideal electron gas in two limit cases of weak ($n_e\Lambda_e^3 \ll 1$) and strong ($n_e\Lambda_e^3 \gg 1$) electron degeneracy (see [11,13] for details). In turn, the high-frequency asymptotics of the spectral energy distribution of equilibrium radiation in an ideal electron gas is characterized by a power-law reduction with increasing frequency at arbitrary degeneracy, and its value is defined completely by electron-gas density $n_e(T,\mu_e)$ as a function of temperature T and chemical potential μ_e.

We note in conclusions that, in spite of the „slowness“ of reduction with increasing frequency (compared to the Planck distribution), the mean energy of equilibrium radiation per unit volume is a finite quantity. It should be stressed that the obtained results may be regarded as exact ones. The thing is that photon energy $\hbar\omega$ in the considered high-frequency asymptotics is assumed to be well above the typical energy values characterizing matter (including the typical energy of Coulomb interaction between charged particles in matter). This implies that the Wien law, which follows directly from the Planck distribution for a blackbody, does not hold in the high-frequency limit in the considered material medium. The equilibrium radiation of a blackbody itself is the result of interaction of charged particles that surround the void filled with radiation. However, the material medium, which is an aggregate of the void with radiation (and, presumably, no particles) and the surrounding matter with radiation, is an inhomogeneous system. Owing to the locality of the Hamiltonian of interaction between the electromagnetic field and charged particles, the mean energy of this interaction in an inhomogeneous system of this kind is proportional to interaction surface area S.

Following the transition to the thermodynamic limit ($V \to \infty$, $S \to \infty$), the surface contribution to the mean energy of the electromagnetic field becomes negligible compared to the bulk contribution, and the Planck distribution for a blackbody is obtained as a result. In the present study, we consider a uniform and isotropic material medium with no isolated voids. Therefore, the inclusion of interaction between particles and the field produces a correction to the Planck distribution, which is regarded as the „zeroth approximation" with the surface effects neglected. This analysis also yields a correct result only in the thermodynamic limit.

At the same time, consistent models for WDM include bound states of electrons localized in the vicinity of nuclei producing WDM ions as compound particles. WDM is commonly associated with a combination of strongly bound ions and moderately degenerate electrons. In this light, further development of the above results should involve the construction of models that take the effects of interparticle interaction into account when characterizing the transverse permittivity. In contrast to the well-known methods used to characterize the longitudinal permittivity [18,19], correct approaches to the analysis of the transverse permittivity should give consistent consideration to the electron intrinsic magnetic moment (see [20] for details).

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Conflict of interest

The author declares that he has no conflict of interest.

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