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Small amplitude breather of the nonlinear Klein–Gordon equation

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A technique for obtaining an approximate breather solution of the Klein–Gordon equation is presented. A breather solution of the equation describing the propagation of nonlinear waves in a graphene-based superlattice is investigated.

Keywords: Klein–Gordon equation, traveling breather, approximate solution, correlation coefficient.

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Introduction

Equation of type

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + F(u) = 0, \tag{1}$$

where $F(u)$ is an odd function, occurs in various branches of theoretical and mathematical physics. A linear approximation of this equation is known in quantum theory as the Klein–Gordon equation, and one of the most important (and most common) special cases of this equation is equation sine-Gordon (SG) [1–3]:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0. \tag{2}$$

The SG equation is remarkable in that it has solutions in the form of solitary waves — solitons and breathers. The exact solution of equation (2) in the form of a travelling breather has the form [4–6]:

$$u_\omega(x,t) = 4\arctan\left(\frac{\sqrt{1-\omega^2}}{\omega} \times \frac{\cos\left(\omega\gamma t - \omega x\sqrt{\gamma^2-1}\right)}{\cosh\left(\sqrt{1-\omega^2}\left(\gamma x - t\sqrt{\gamma^2-1}\right)\right)}\right). \tag{3}$$

Here $\gamma = (1-V^2)^{-1/2}$, V is group velocity of impulse propagation.

The work [2] is devoted to the numerical study of kink and breather solutions of the sine-Gordon equation under the influence of „force“determined by the Heaviside function $H(\xi)$:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = F(x,t), \tag{4}$$

$$F(x,t) = AH(t-x).$$

Boundary conditions of various types are considered, and the exact solutions of equation (2) are used as the initial condition for solving the perturbed equation (4).

In the paper [3] solutions of the two-dimensional sine-Gordon equation, similar to kink solutions of the one-dimensional equation (2) are studied:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \sigma\frac{\partial^2 u}{\partial y^2} + \sin u = 0, \tag{5}$$

$\sigma = \pm 1$. Here, the interaction of a kink and an antikink is considered, and a system of equations is derived that makes it possible to determine the width and shape of both a lone kink and soliton-like pulses interacting with each other. A numerical procedure for determining the kink center, and a variational procedure for studying the shape dynamics of a single kink in the direction perpendicular to its propagation are discussed. The paper [7] also relates to the numerical study of stationary and traveling breather solutions of the two-dimensional sine-Gordon equation.

The paper [8] relates to the study of real space-periodic, central-symmetric solutions of the Klein–Gordon equation of the form:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + m^2 u - \Gamma(x)u^3 = 0. \tag{6}$$

The author of the paper [8] calls the obtained solutions breathers by analogy with solutions of the sine-Gordon equation with similar properties. In [8] the equation (6) was reformulated as a system of coupled nonlinear Helmholtz equations under certain field conditions in the far zone.

Exact and approximate breather solutions of various options of the Klein–Gordon equation are studied in a number of papers, for example, [9,10]. The paper [9] considers the numerical solution of the discrete Klein–Gordon equation, which describes oscillations of an infinite chain of particles coupled with nearest neighbors, in a local potential V [11]:

$$\frac{d^2x_n}{dt^2} + V'(x_n) = \gamma(x_{n+1} + x_{n-1} - 2x_n). \tag{7}$$

In the long-wavelength approximation the equation (7) is reduced to the Klein– Gordon equation (1).

In the paper [11] related to the study of oscillation in a one-dimensional chain of atoms considering anharmonicity, a method for the approximate solution of the Klein–Gordon equation in the limit of small amplitudes ($u \ll 1$) is proposed. In this case, equation (1) is transformed to the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u - \beta u^3 = 0. \tag{8}$$

In papers [11,12] a one-parameter localized periodic solution is sought in the form

$$u = A(x)\cos(\omega t) + B(x)\cos(3\omega t) + \dots . \tag{9}$$

For the solution (9) convergence it is necessary to set $|A| \gg |B| \dots$. In the case of small amplitudes, the largest contribution to the solution is provided by the first term, i. e., solution is like a standing wave. Such solutions are sometimes referred to in the literature as standing (stationary) breathers. For example, for the sine-Gordon equation the solution [4] is known:

$$u_\omega(x,t) = 4\arctan\left(\frac{\sqrt{1-\omega^2}}{\omega}\,\frac{\sin(\omega t)}{\cosh(x\sqrt{1-\omega^2}}\right), \tag{10}$$

which for values $\omega \approx 1$ can be approximately represented as $y = A(x)\cos\omega t)$. The purpose of this paper is to obtain solution in the form of a travelling breather for the nonlinear Klein–Gordon equation (8) using the method developed in [11,12].

1. Method of obtaining an approximate solution of the nonlinear Klein–Gordon equation in the form of a small-amplitude travelling breather

Consider obtaining the approximate analytical solution of the Klein–Gordon equation (1). As noted, in case of $u \ll 1$ the equation (1) is transformed to the form (8). We will look for solution to equation (8) in the form of a series (9) with uniformly decreasing coefficients in front of the cosines. Substituting (9) into (8) and taking into account that $|A| \gg |B|$ (for convergence of the series (9)), we obtain the system

$$\begin{cases} \frac{d^2A}{dx^2} - (1-\omega^2)A = -\frac{3}{4}\beta A^3, \\ \frac{d^2B}{dx^2} + (9\omega^2 - 1)B = -\frac{1}{4}\beta A^3. \end{cases} \tag{11}$$

By solving the system in the region of localized solutions limited at infinity, it is possible to determine the functional form of the small-amplitude breather.

Let's try to develop, by analogy with the above, a method for finding traveling low-amplitude breathers that are the solution of equation (8). In fact, the solution of interest to us is now a two-parameter delocalized time-periodic solution. We will proceed from the well-known form of such solution for the sine-Gordon (2) equation represented by expression (3). We will look for solution to equation (8) in the form

$$\begin{aligned} u &= A\Big(\sqrt{1-\omega^2}\big(\gamma x - t\sqrt{\gamma^2-1}\big)\Big) \\ &\quad\times \cos\big(\gamma\omega t - \sqrt{\gamma^2-1}\,\omega x\big) \\ &\quad+ B\Big(\sqrt{1-\omega^2}\big(\gamma x - t\sqrt{\gamma^2-1}\big)\Big) \\ &\quad\times \cos\big(3\gamma\omega t - 3\sqrt{\gamma^2-1}\,\omega x\big) + \dots . \end{aligned} \tag{12}$$

Now, following the previous method, after substituting (12) into equation (8) and some manipulations, we obtain a system for determining functions $A(\xi)$, $B(\xi)$, (next we implement the notation $\xi = \sqrt{1-\omega^2}(\gamma x - t\sqrt{\gamma^2-1})$):

$$\begin{cases} (1-\omega^2)\frac{d^2A}{d\xi^2} - (1-\omega^2)A = -\frac{3}{4}\beta A^3, \\ (1-\omega^2)\frac{d^2B}{d\xi^2} + (9\omega^2 - 1)B = -\frac{1}{4}\beta A^3. \end{cases} \tag{13}$$

The solution of the first equation in (13) that suits us in terms of properties is the function

$$A(\xi) = \left(\frac{8}{3\beta}(1-\omega^2)\right)^{1/2}\frac{1}{\cosh(\xi)}. \tag{14}$$

For the sine-Gordon equation $\beta = 1/6$ and in the highest approximation we obtain the solution

$$u = 4\sqrt{1-\omega^2}\,\frac{\cos(\gamma\omega t - \sqrt{\gamma^2-1}\,\omega x)}{\cosh\big(\sqrt{1-\omega^2}(\gamma x - t\sqrt{\gamma^2-1})\big)}. \tag{15}$$

It can be seen from (3) that, in fact, the condition $u \ll 1$ means $\sqrt{1/\omega^2 - 1} \ll 1 \Longrightarrow \omega \approx 1$. Considering this the approximate solution obtained by expanding (3) into a series takes the form

$$u = 4\sqrt{\frac{1}{\omega^2} - 1}\,\frac{\cos(\gamma\omega t - \sqrt{\gamma^2-1}\,\omega x)}{\cosh\big(\sqrt{1-\omega^2}(\gamma x - t\sqrt{\gamma^2-1})\big)}. \tag{16}$$

and coincides with (15).

2. Example — approximate breather solution of the equation for the vector-potential of an electromagnetic field in graphene superlattice

As an example, we will consider the equation describing the propagation of sole electromagnetic waves in the graphene superlattice (GSL) [13–15]:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \frac{\omega_0^2 b^2 \sin u}{\sqrt{1 + b^2(1-\cos u)}} = 0. \tag{17}$$

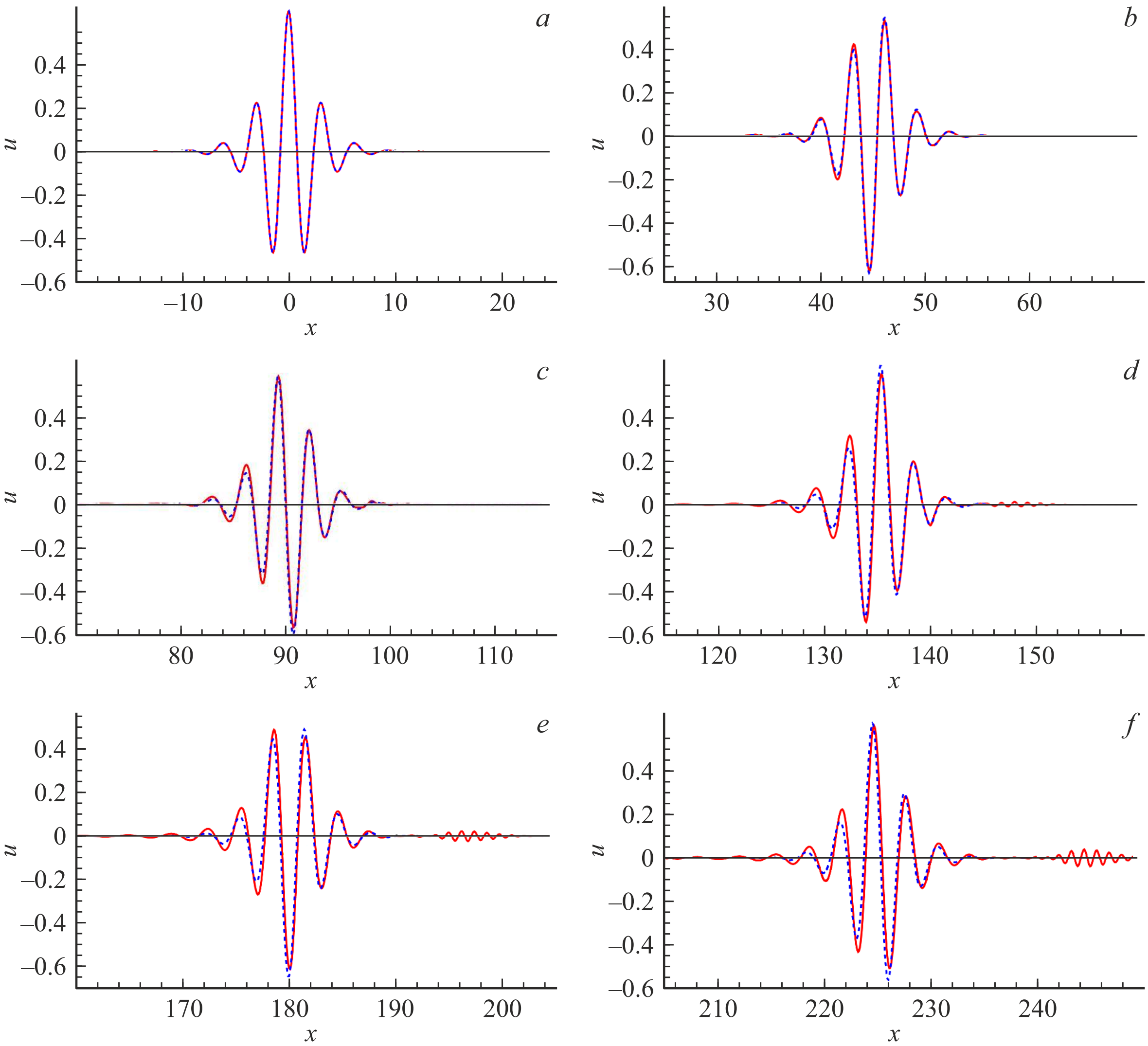

Figure 1. Comparison of numerical (red line (online version)) and approximate analytical (dashed blue line (online version)) at different times, t: a — 0; b — 50; c — 100; d — 150; e — 200; f — 250.

Here $u = edA_z/\hbar c$ is the dimensionless component of the vector potential in the direction of alternating SL layers, c is the speed of light, e is the elementary electric charge, $\omega_0^2 = 2\pi n_0 e^2 d^2 \Delta/(a_0 \hbar^2)$, n_0 is surface concentration of charge carriers, $a_0 = 0.12$ nm is graphene layer thickness, $b = \Delta_1/\Delta$, the parameters Δ and Δ_1 can be conventionally called the half-widths of the forbidden and allowed mini-band, respectively, d is the SL period. It is assumed that the layers alternate along the z axis. Equation (17) has 2π-impulse solution, expressed implicitly [13]:

$$\int_{\pi}^{u(\xi)} \frac{du}{\sqrt{\sqrt{1 + b^2(1 - \cos u)} - 1}} = 2\xi, \tag{18}$$

$\xi = (x - vt)/L_0$, $L_0 = (c/\omega_0)\sqrt{1 - v^2/c^2}$ v is electromagnetic pulse speed.

Let's transform equation (17). Let's introduce new variables

$$x\omega_0 b/c \to x, \quad t\omega_0 b \to t. \tag{19}$$

Equation (17) takes the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\sin u}{\sqrt{1 + b^2(1 - \cos u)}} = 0. \tag{20}$$

Expanding $\sin u/\sqrt{1 + b^2(1 - \cos u)}$ into u series up to cubic terms, we obtain

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u - \left(\frac{b^2}{4} + \frac{1}{6}\right) u^3 = 0, \tag{21}$$

then

$$\beta = \frac{b^2}{4} + \frac{1}{6}. \tag{22}$$

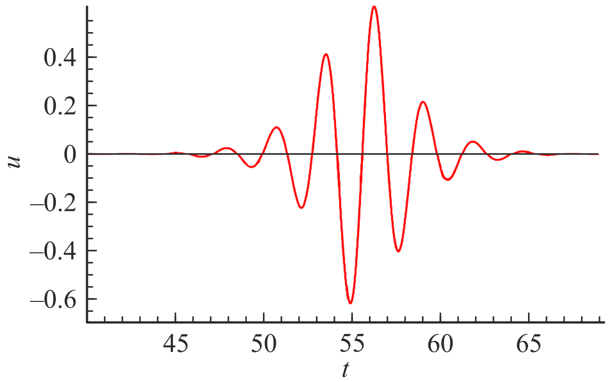

Figure 2. Approximate solution vs. time, $x = 50$.

According to (12), (14), the solution of equation (20) in the form of breather has the form

$$u = \left(\frac{32(1-\omega^2)}{3b^2+2}\right)^{1/2} \frac{\cos(\gamma\omega t - \omega x\sqrt{\gamma^2-1})}{\cosh\left(\sqrt{1-\omega^2}(\gamma x - t\sqrt{\gamma^2-1})\right)}. \tag{23}$$

Passing to the original notations, we obtain

$$u = \left(\frac{32(1-\omega^2)}{3b^2+2}\right)^{1/2} \times \frac{\cos(t\,\frac{\gamma\omega}{(\omega_0 b)} - x\,\frac{\omega c}{\omega_0 b}\sqrt{\gamma^2-1})}{\cosh\left(\sqrt{1-\omega^2}\left(x\,\frac{\gamma c}{\omega_0 b} - \frac{t}{\omega_0 b}\sqrt{\gamma^2-1}\right)\right)}. \tag{24}$$

The study of the stability of the approximate solution (23) is of interest. Using the Wolfram Mathematica package, we will numerically solve equation (20), taking function (23) as the initial condition. Fig. 1 shows graphs of the approximate analytical and numerical solutions at different times. When plotting graphs, it was assumed that $b = 0.90$, $\omega = 0.97$. It can be seen from the graphs in Fig. 1 that although the condition $|u| \ll 1$ is not satisfied for these values, the numerical solution turns out to be close to the analytical one presented above and exhibits stability, i. e. the range of applicability of the approximate analytical solution turns out to be somewhat wider than originally assumed.

Fig. 2 shows the solution vs. time for a fixed x. It can be seen from this Figure that the pulse width is approximately 20 units along the time axis. Fig. 3 shows a graph of the solution when time t is shown along one of the axes, and the spatial coordinate x is shown along the other axis.

To quantify the differences between the numerical solution and the approximate analytical solution, we use the following method. For a given t in the region where the solution takes non-zero solutions (i.e. $|u|$ is greater than some small positive value ε), we calculate the extrema of the numerical solution. Based on the list of values $\{x_i, |u(x_i, t)|\}$, where x_i determine the position of the extrema of the numerical solution, we plot the interpolation function. Fig. 4 shows graphs of the interpolation function, i. e., the envelope of the absolute value of the numerical solution near the amplitude maximum at different times. After finding the position $x_{\max}(t)$ of the maximum of the interpolation function, we determine the segment $[x_{\max}(t) - L, x_{\max}(t) + L]$, where L is half the pulse width in space (as per Figs 1, 4, $L \approx 10$). Next, randomly select N values $\{x_i\}_{i=1...N}$ from this segment and form two vectors: $a = \{u_{appr}(x_i)\}_{i=1...N}$ and $b = \{u_{num}(x_i)\}_{i=1...N}$ — values of approximate analytical and numerical solutions, respectively, at the points $\{x_i\}_{i=1...N}$. To compare the approximate analytical and numerical solutions we calculate the correlation coefficient at different times:

$$K_{corr} = \sum_{i=1}^{N} \frac{(a_i - \overline{a})(b_i - \overline{b})}{\sigma_a \sigma_b (N-1)}. \tag{25}$$

Here $\overline{a} = \sum_{i=1}^{N} a_i/N$ is average random value, $\sigma_a = \sqrt{\sum_{i=1}^{N}(a_i - \overline{a})^2/(N-1)}$ -is standard deviation. Fig. 5 shows the correlation coefficient between the numerical solution and the approximate analytical solution vs. time. It can be seen that the correlation coefficient monotonically decreases, however, even at the moment $t = 250\,K_{corr} \approx 0.94$, which indicates that the proposed approximate solution in the form of travelling breather decays rather slowly.

3. Comparison with the papers of other authors. Discussion of results

The paper [16] considers a two-component breather solution of the nonlinear Klein–Gordon equation

$$\frac{\partial^2 U}{\partial t^2} - C\frac{\partial^2 U}{\partial x^2} = -\alpha_0^2 U + \frac{\alpha_0^2}{6}U^3, \tag{26}$$

obtained by expanding in Taylor series the right side of the sine-Gordon equation for $U \ll 1$. In [16] the generalized pertubative reduction method developed in [17–19] is used. The solution obtained in [16] is

$$U(x,t) = A\,\mathrm{sech}\big((t - x/V_0)/T\big)\big\{\cos(k_1 x - \omega_1 t) + B\cos(k_2 x - \omega_2 t)\big\}. \tag{27}$$

Solution (26) is similar in form to solutions (15), (23) obtained in this paper. The essence of the perturbation reduction method used in [16–19] is the search for a solution to a nonlinear partial differential equation in the form of an amplitude modulated plane wave:

$$U = \sum_{\alpha=1}^{\infty} \varepsilon^\alpha U^{(\alpha)},$$

$$U^{(\alpha)} = \sum_{l=-\infty}^{+\infty} U_l^{(\alpha)}(\tau, \xi)\exp[il(kx - \omega t)], \tag{28}$$

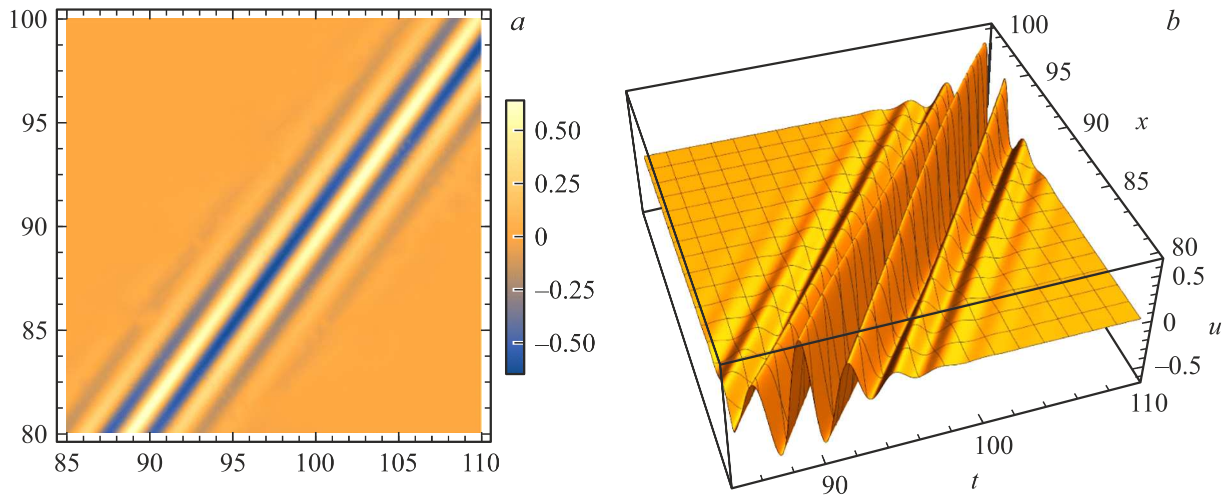

Figure 3. General view of the approximate solution in the region $t = 85{-}100$, $x = 80{-}100$: *a* — heat map; *b* — 3D graph.

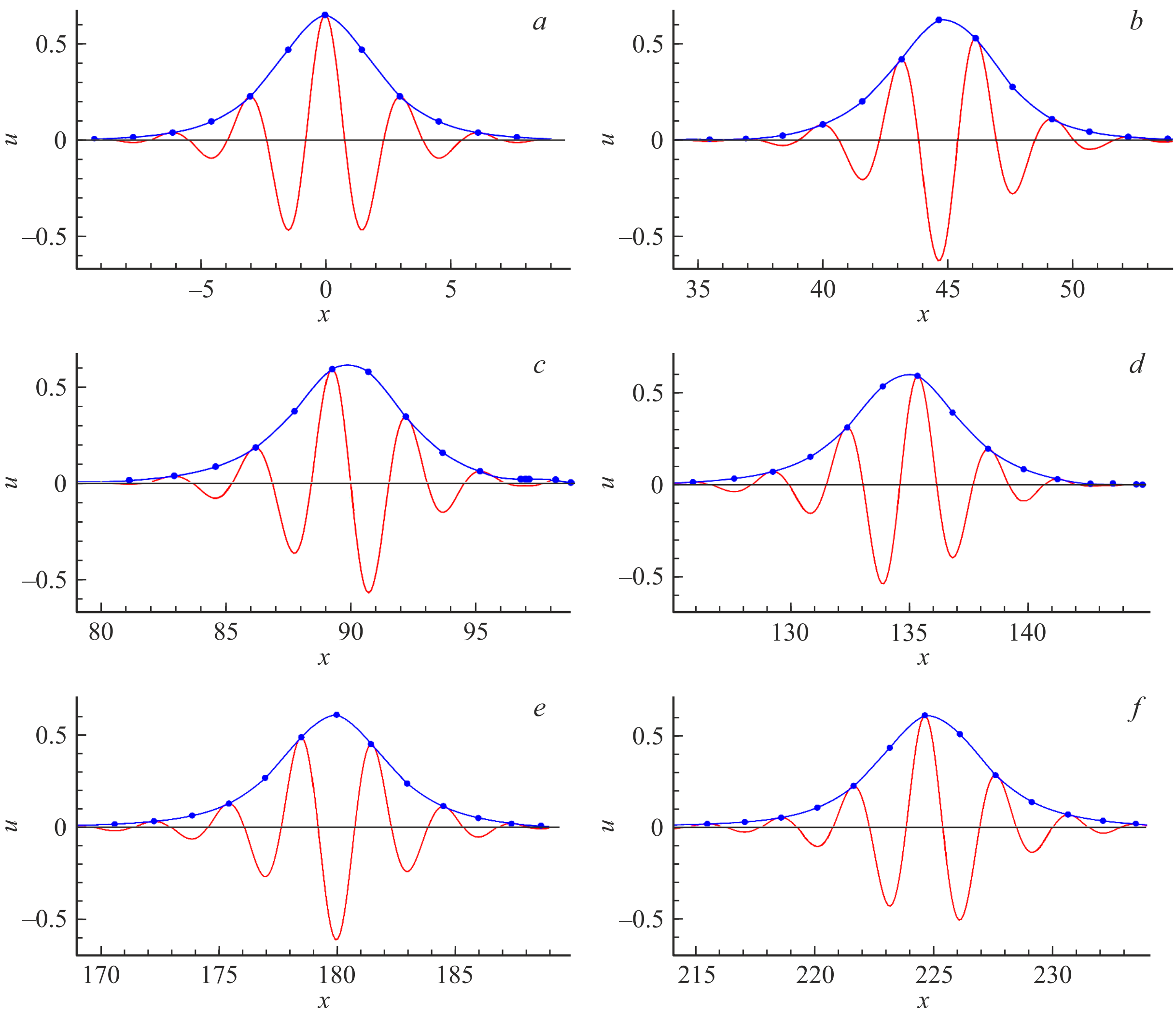

Figure 4. Graphs of the interpolation function, i. e. the envelope of the absolute value of the numerical solution (blue line (in the online version) and dots) near the maximum at different times, *t*:*a* — 0; *b* — 50; *c* — 100; *d* — 150; *e* — 200; *f* — 250.

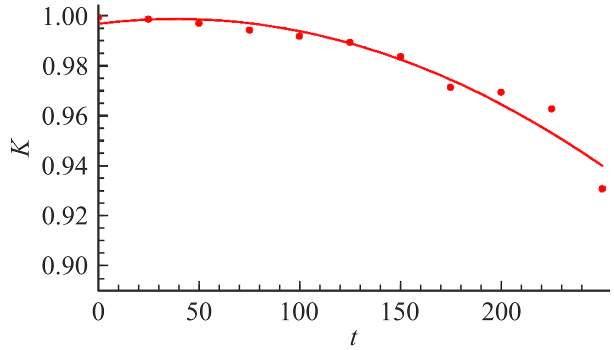

Figure 5. Correlation coefficient between the numerical and approximate analytical solution vs. time.

where ε is some small parameter, $\tau = \varepsilon^2 t$, $\xi = \varepsilon(x - \lambda t)$, $\lambda = \partial\omega/\partial k$ is group velocity, $U_l^{(\alpha)} = U_{-l}^{(\alpha)*}$. Thus, the authors [17], solving the equation of the form (26) for arbitrary values of the coefficients in front of the linear and cubic terms on the right-hand side, obtain a nonlinear Schrodinger equation for the modulating function. However, as shown in [18], the solution of equation (26), taken in the form of the amplitude modulated plane wave, turns out to be unstable. From our point of view, the method proposed in this paper for obtaining the approximate solution of the nonlinear Klein–Gordon equation in the form of small-amplitude breather has an advantage over the perturbation reduction method used in [16–19] due to its simplicity.

The paper [15] relates to the study of breather solutions of equation (17), which describes the propagation of nonlinear waves in a graphene superlattice. In [15] the inelastic collision of the kink and the antikink, described by expression (18), moving with the same magnitude and oppositely directed velocities, which are the solution of equation (17), is numerically studied. The calculation shows that after the collision the solitary waves continue to move up to „infinity"if their speed is greater than some critical value or belongs to a set of resonant windows. Otherwise, after the collision, the solutions form a state similar to the breather, which slowly decays, radiating energy. The fractal structure of these resonant windows is characterized by a multi-index notation, the main features of this structure are compared with the predictions of the theory of resonant energy exchange and show good agreement with this theory.

Conclusion

Thus, the paper proposes a method for obtaining the approximate solution of the nonlinear Klein–Gordon equation, which is the small-amplitude traveling breather. The example of obtaining such solution for the equation describing the propagation of nonlinear waves in the graphene superlattice is considered. The obtained solution was analyzed for stability. It is shown that the form of the solution changes weakly over a time interval of tens of pulse widths.

Conflict of interest

The authors declare that they have no conflict of interest.

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