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Approximation of the solution of an internal problem of electrodynamics by the method of eigenfunctions

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The problem of constructing approximation models of radiating and reradiating structures based on the eigenfunction method is considered. The ambiguity of the solution of this problem is noticed. Techniques are proposed which can be used in developing approximation models. An expression is presented for estimating the error introduced by the approximation model. As an example, one of the possible options for constructing an approximation model for a tubular antenna is given. The results of numerical simulation are presented.

Keywords: method of eigenfunctions, method of moments, singular integral equations, approximation model, tubular antenna.

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Nowadays, the problems of radiation and diffraction of electromagnetic waves are often solved by using computer-aided design systems. One of the most efficient methods used in creating those systems is the finite-element method [1]. Therewith, using the method of moments [2] it is possible to reduce the internal problem of electrodynamics to a set of linear algebraic equations (SLAE) whose direct solution complexity may be assessed, depending on its dimension N, as $O(N^3)$. Dimension N is directly associated with the choice of the projection function system (PFS). PFS may be regarded as optimal if it converts the SLAE matrix to the diagonal form. In paper [3] it was proposed to solve this problem by using the eigenfunction method (EFM). As an alternative to EFM, the characteristic-mode method may be regarded [4]. In addition, paper [3] considered advantages and disadvantages of the mentioned methods; for example, solution by EFM of the problem of diffraction on a helical particle was presented. There was made a conclusion that EFM may become a good basis for creating approximation models (AM) for solving the internal problem of electrodynamics. This paper proposes some techniques to be used in creating AMs. As an example, one of the options for constructing AM for a tubular antenna is considered.

Typically, the problem of source identification (the internal problem of electrodynamics) gets reduced to an integral equation (IE) that may be presented in the operator form:

$$\mathrm{A}(M)u = v. \tag{1}$$

Here A is the integral operator, M is the kernel, v is the IE right-hand part. EFM implies representation of the equation (1) solution as a sum of the approximated solution u' and residue r:

$$u = u' + r, \quad u' = \sum_{j=1}^{N} u'_j \phi_j,$$

$$u'_j \in \mathbf{U}' = \hat{\mathbf{W}}'\mathbf{V}, \qquad \hat{\mathbf{W}}' = (\hat{\mathbf{J}}\hat{\mathbf{X}}^{-1}\hat{\mathbf{J}}^T)'. \tag{2}$$

Here $\hat{\mathbf{U}}'$ is the approximation of the SLAE solution $\hat{\mathbf{M}}\hat{\mathbf{U}} = \hat{\mathbf{V}}$ obtained by the method of moments; $\hat{\mathbf{V}}$ is its right part; $\hat{\mathbf{J}}$ and $\hat{\mathbf{X}}$ are the matrices of eigen vectors (EV) and eigen numbers (EN) which are the solution of spectral problem $\hat{\mathbf{M}}\hat{\mathbf{J}} = \hat{\mathbf{X}}\hat{\mathbf{J}}$; ϕ_j are the moment-method projection functions. Matrix $\hat{\mathbf{X}}$ is diagonal; therewith, $\xi_{j,j} \equiv \xi_j \in \mathbf{X}$ are the $\hat{\mathbf{M}}$ eigen numbers. Columns $\mathbf{J}_j$ of matrix $\hat{\mathbf{J}}$ represent EVs corresponding to ENs ξ_j. The prime is used to denote truncation of initial matrices $(j = 1 \dots P,\ P < N)$. The prime after brackets means its application to all the objects in the brackets. Truncation increases r in (2) by an in-advance known value.

Let it be that $\mathbf{p} = \{x, \gamma_1, \gamma_2 \dots\}$ is the point belonging to the approximation region A, $x = h/\lambda$, h is the basic geometric parameter of the system under study, $\gamma_1, \gamma_2 \dots$ are its remaining parameters, λ is the wavelength. Physically, elements $\hat{\mathbf{J}}$ and $\hat{\mathbf{X}}$ are smooth functions of coordinates $\mathbf{p}$. Their approximations $\hat{\mathbf{J}}''$ and $\hat{\mathbf{X}}''$ for $\mathbf{p} \in \mathrm{A}$ provide an AM describing the system state in the specified region with a preset accuracy. The approximation relative error may be estimated using the following relation:

$$\rho_A = \rho_A(\mathbf{p}) = |\hat{\mathbf{W}}''\hat{\mathbf{M}}' - \hat{\mathbf{I}}^*|/|\hat{\mathbf{I}}^*|, \ \ \hat{\mathbf{W}}'' = (\hat{\mathbf{J}}\hat{\mathbf{X}}^{-1}\hat{\mathbf{J}}^T)'',$$

$$\hat{\mathbf{M}}' = (\hat{\mathbf{J}}\hat{\mathbf{X}}\hat{\mathbf{J}}^T)', \quad (\hat{\mathbf{J}}\hat{\mathbf{J}}^T)' = \hat{\mathbf{I}}^*. \tag{3}$$

Approximations $\hat{\mathbf{J}}''$ and $\hat{\mathbf{X}}''$ are performed based on initial arrays $\{j\}$ and $\{\xi\}$ which are $\hat{\mathbf{J}}'$ and $\hat{\mathbf{X}}'$ calculated for a point array $\{p\} \in \mathrm{A}$. Restoration of $\hat{\mathbf{J}}'$ and $\hat{\mathbf{X}}'$ from $\{j\}$ and $\{\xi\}$ may be performed using different interpolation options (linear, spline-interpolation, etc.). The $\{j\}$ and $\{\xi\}$ dimensions are rather large; therefore, AM implies formation of procedures T_j and T_ξ allowing restoring $\{j\}$ and $\{\xi\}$ with a known accuracy from more compact underlying arrays $\{j^*\}$ and $\{\xi^*\}$, respectively. Desirable is that $\{j^*\}$ and $\{\xi^*\}$ have a satisfactory compaction potential defined by the number of elements whose elimination from the arrays would not initiate a significant increase in error of the $\{j\}$ and $\{\xi\}$ restoration. In paper [5] we showed that the frequency dependence of the EN and EV elements is well approximatable by polynomial expansions. The compaction potential of $\{j^*\}$ can be increased by projecting the instantaneous-frequency EV on fixed-frequency EV.

Consider one of the versions of constructing such an AM for tubular antenna (Fig. 1) whose internal problem is defined in the form of a singular IE [6]. In Fig. 1, the following designations are used: L is the antenna length, a is the tube radius, $2b$ is the clearance width, s is the clearance center displacement along axis Oz. The IE form is consistent with problem (1) where $u = \eta(z)$ is the surface current density, $v = E_z^{ext}(z)/Z$; E_z^{ext} is the z-component of the impressed field, Z is the wave resistance of the medium. For the structure under consideration, $\mathbf{p} = \{x, \gamma\}$ where $x = L/\lambda$, $\gamma = a/L$. Due to the symmetry in the framework of the method of moments it is possible to construct two independent SLAEs with respect of even and odd PFSs. In our case, the antenna was divided into equal-length segments; as the basis and test ones, superpositions of piecewise-constant functions and delta-functions accounting for the structure symmetry were used respectively. As initial arrays, the following were taken:

$$\{\xi\} = \{\xi_{r,f,p}\} = \xi'_p(\mathbf{p}_{f,r}), \qquad \{j\} = \{j_{r,f,n,p}\} = \mathbf{J}'(\mathbf{p}_{f,r}),$$

$$\mathbf{p}_{f,r} = \{x_f, \gamma_r\}, \quad x_f \in \{x\}, \quad \gamma_r \in \{\gamma\},$$

$$n = 1 \dots N_s/2, \qquad p = 1 \dots P,$$

$$f = 1 \dots N_f, \qquad r = 1 \dots N_r, \tag{4}$$

N_f is the number of wavelengths, N_r is the number of normalized radii, N_s is the number of segments. Procedure T_ξ determines array $\{\xi^*\} = \{\xi^*_{r,c,p}\}$ whose c-elements are coefficients of the $\{\xi\}$ f-elements expansion in terms of Chebyshev polynomials ($c = 1 \dots N_c$). Procedure T_j determines array $\{j^*\} = \{j^*_{r,c,q,p}\}$ whose c-elements are coefficients of the array $\{j'_{r,f,q,p}\} = \sum_n \{j_{r,f,n,q}\}\{j_{r,1,n,p}\}$ f-elements expansion in terms of Chebyshev polynomials.

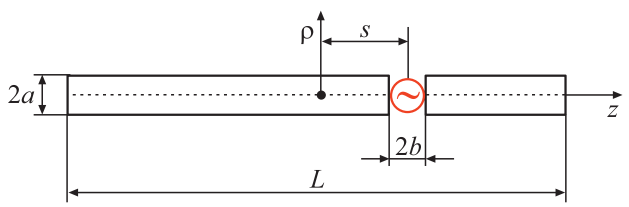

Figure 1. The tubular antenna geometry.

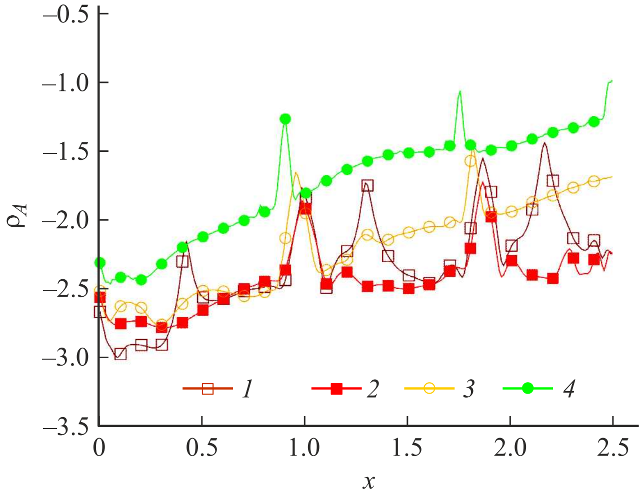

Figure 2. Estimation of the approximation error according to (3). $\gamma = 1.875 \cdot 10^{-4}$ (*1*), $9.375 \cdot 10^{-4}$ (*2*), $9.375 \cdot 10^{-3}$ (*3*), $9.375 \cdot 10^{-2}$ (*4*).

Compaction of arrays $\{\xi^*_{r,c,p}\}$, $\{j^*_{r,f,q,p}\}$ consists in eliminating low-amplitude elements so that related residuals ρ_ξ and ρ_j between the initial and compacted arrays do not exceed the in-advance known values of ε_ξ and ε_j.

The AM was constructed using the following parameters: $N_s = 300$, $N = 150$, $P = 25$, $N_c = 64$, $2b/L = 1/100$, $\varepsilon_\xi = 10^{-7}$, $\varepsilon_j = 10^{-3}$, $x_f \in [10^{-3}; 2.5]$, $\gamma_r \in [10^{-4}; 10^{-1}]$. In calculations, double-precision numbers were used. For x, a uniform step (250 points) was used; for γ, the range was divided into three decades $[1; 10] \cdot 10^{n-5}$, where $n = 1, 2, 3$ is the decade No. Inside each decade, the arrays were calculated at the following points: $\gamma/10^{n-5} = 1$, 1.375, 1.75, 2.5, 3.75, 5, 6.25, 7.5, 8.75, 10 (three decades contain four boundary points and eight inner points each). Thus, the total number of computational points for γ was 28. The calculations provided the following real dimensions of the initial and compacted underlying arrays: $\{\xi\}$ — 2.6 MB, $\{\xi'\}$ — 106 KB, $\{j\}$ — 3.36 GB, $\{j'\}$ — 1 MB, dimension of the underlying array $\{j_{r,1,n,p}\}$ was 2 MB. Fig. 2 presents the results of calculating the error of the inverse integral operator approximation via formula (3). Fig. 3 presents normalized curves representing approximated distributions of the surface current density for the worst case, which were calculated using the model presented here. The antenna was assumed to be symmetrical ($s = 0$), while field E_z^{ext} was assumed to be localized in the vicinity of the clearance. Similar curves obtained by using directly calculated ENs and EVs exhibit a quite small visual difference from the curves obtained using the AM.

Thus, this paper is devoted to the problem of constructing AM of radiating and reradiating structures based on the eigenfunction method. General principles of the AM creation are considered. Notice that the in-depth analysis of properties of the underlying arrays makes it possible to improve the initial AM. The initial AM may

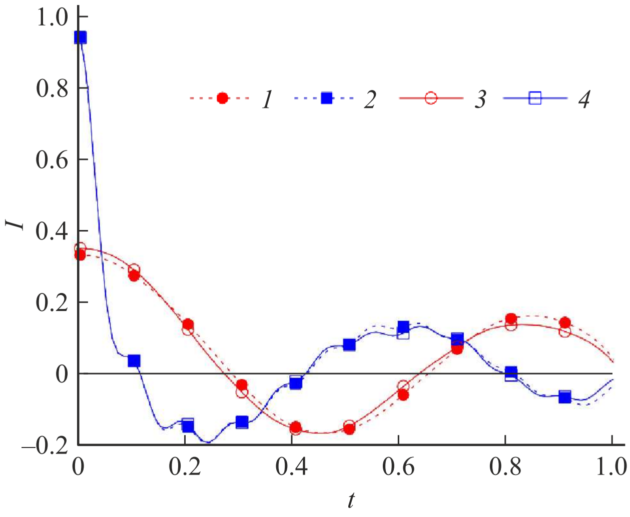

Figure 3. Comparison of normalized current distributions obtained based on the initial ($I' = \eta'/\eta_{\max}$) and restored in the framework of AM ($I'' = \eta''/\eta_{\max}$) eigen numbers and eigen vectors. *1* — ReI', *2*— ImI', *3* — ReI'', *4* — ImI''. $\varepsilon_\xi = 10^{-7}$, $\varepsilon_j = 10^{-3}$, $x = 2.5$, $\gamma = 9.375 \cdot 10^{-2}$ (the worst case), $I_{\max} = \max(|\eta'(t)|, |\eta''(t)|)$, $t = 2z/L$.

become a basis for a more compact AM with a lower approximation accuracy. The paper presents an easily implementable version of constructing a tubular antenna AM for a specified frequency range; in this version, the tube radius is taken into account. The authors believe that this AM may be used in engineering calculations and for creating models of multi-component vibrator antennas. A drawback of the proposed AM is some loss of physical sense of data stored in the underlying arrays. In [3], we emphasized especial importance of extrema of the EN frequency dependences, which define the general character of the internal problem solution. Ideally, such information should be included in the underlying arrays in the explicit form. This allows maximally accurate storage of data necessary for analyzing, interpreting and predicting the obtained results. The proposed AM contains these data implicitly, and obtainment of information on the extrema needs additional (relatively simple) computation procedures with somewhat less accuracy. In future the authors plan to create AMs accounting for this aspect to the fullest extent.

Conflict of interests

The authors declare that they have no conflict of interests.

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