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# Spin structure and spin magnetic susceptibility of two-dimensional Wigner clusters

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Spin states of two-dimensional Wigner clusters are considered at low temperatures, when all electrons are in ground coordinate states. It is assumed that the behavior of the spin subsystem is determined by antiferromagnetic exchange integrals. The spin states in such a system in the presence and without a magnetic field are described in terms of the Ising model. The spin structure, correlation function, and magnetic susceptibility of the cluster are found by computer simulations. It is shown that the spin susceptibility experiences oscillations in the magnetic field, owing to the magnetically induced rearrangements of the spin subsystem.

**Keywords:** Wigner cluster, Ising model, spin structure, spin correlation function, spin magnetic susceptibility.

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## 1. Introduction

Two-dimensional Wigner lattices were the subject of study since the 1970s [1–3]. In these systems, the Coulomb repulsion is compensated by opposite charges separated by an insulating layer from the two-dimensional quantum well. Classical electrons occupy potential energy minima, and quantum electrons form wave functions near the minima. If the quantization energy becomes large, the electron lattice melts, and the system transforms into an electron gas.

Various properties of two-dimensional Wigner lattice (2DWL) were studied, in particular interaction energy, plasmon spectrum, melting phase transition and the magnetic field influence. Two-dimensional Wigner clusters (2DWC) with a small number of electrons, where electrons are kept from scattering by an external potential along the surface, were also considered. In [4] the spectrum of electronic oscillations was studied, and in [5,6] — two-stage melting of 2DWC was studied. Many articles, including reviews, are devoted to the study of 2DWC [6–15].

Theoretically, 2DWL and 2DWC were considered classically and quantum not considering spins. At the same time, the exchange interaction is weaker than the Coulomb interaction. However, in the case of small differences between the ordering minima, the exchange interaction can affect the ordering of the coordinates.

Fine rearrangement of 2DWC states using a confining potential and a magnetic field leads to their possible application in electronics [16], lasers [17,18] and quantum calculations [19].

A class of systems should be mentioned, namely two-dimensional lattices of ions trapped on the surface of liquid helium (see, for example [20]). In these systems the effective mass is much larger than the electron mass, which promotes the ion condensation.

The experimental implementation of 2DWCs in semiconductors has encountered serious difficulties due to the low required electron density, where disorder usually overcomes the e-e interaction. However, in recent papers such systems were implemented in the form of a one-dimensional Wigner cluster with the transition „string–zigzag“ [21,22], a Wigner crystal in a monolayer semiconductor [23] and two-layer Wigner crystals in heterostructures of transition metal dichalcogenides [24].

Electrons are usually considered as structureless particles. The coordinate ordering of electrons in the cluster does not directly affect their spin degree of freedom. However, the spin structure of the 2DWC is important for its thermodynamic properties.

The problem of exact diagonalization of the spin Hamiltonian 2DWC in the Heisenberg model requires a lot of time, so it was carried out for small numbers of electrons (see, for example, for a two-dimensional two-center oscillator [8] and a two-dimensional harmonic confining potential [9] for $n = 4$ and for $n \leq 8$ [10]).

There are a number of studies of Coulomb blockade in low-electron quantum dots (see, for example, the review [11]). The Coulomb blockade approach differs from the 2DWC by blurring of the electron density inside the dot. In this approximation, unlike 2DWC, the correlation between electrons is not taken into account or is considered weak and perturbative.

A parabolic quantum dot with up to 20 electrons was studied considering spins and using test wave functions [12].

Confinement strongly affects correlations due to broken translational symmetry, which leads to electron lattice localization. The exact quantum problem for three electrons in the asymmetric parabolic well is considered in [13]. The cumbersome nature of the calculations does not make it possible to significantly increase the number of electrons.

A review of results on electronic states in two-dimensional quantum dots and rings is contained in [14]. Among them there are papers relating classical ordering [6,15]. Classical configurations with geometric shells $(n_1, n_2, \ldots)$ of electrons in 2DWCs with different numbers of electrons were obtained in [6,15,25]. These papers differ in the number of electrons considered. In particular, for $n = 6$ there are configuration (1,5) [15] or (3,3) [6], at $n = 10$ configuration (3,7) [6,26] or (2,8) [15]. These differences arise from small differences in the energy minima corresponding to „isomers" of 2DWC. Isomerism leads to easy melting of the structure at low temperatures.

In [25] we studied the structure of the classical 2DWC in the axially symmetric and asymmetric parabolic potential $k(x^2 + b^2y^2)$. Inside the cluster the electrons form a distorted triangular lattice. The competition between the boundary and internal energies leads to polycrystallinity of the inner part of the cluster and ordered arrangement of boundary electrons, as well as to the presence of topological defects inside the cluster. It turned out that the structures of classical and quantum 2DWC are slightly different. We also considered the rotation of 2DWC under the influence of an alternating magnetic field [26].

Here we will focus on the issue of the spin structure of stationary multielectron clusters. Even if the exchange interaction turns out to be weak compared to the spinless Hamiltonian, the Heisenberg exchange Hamiltonian is a matrix of size $2^n \times 2^n$, the dimension of which grows exponentially with $n$ increasing. In this case, the simpler Ising Hamiltonian significantly simplifies the problem of spin ordering, preserving the main features of the position. The Ising model is the most popular formulation of the theory of phase transitions. 2DWC with large $n$ corresponds to many aspects of the two-dimensional Ising model, but differs in finite size and lattice inhomogeneity. Besides, in relation to 2DWC, the spin order of any $n$ is specific. In 2DWC the lattice is frustrated, finite, and contains polycrystalline blocks and topological defects. 2DWC also has different densities inside and at the boundary, this affects the magnetic properties.

Our goal is to study the spin structure and magnetic moment, as well as the susceptibility of electrons in symmetric and asymmetric parabolic quantum wells. We will consider a multi-electron 2DWC based on classical ordering and then include exchange interaction to study spin ordering based on the Ising model. This consideration of 2DWC is justified by its greater simplicity, which makes it possible to expand the number of electrons considered.

The article is structured as follows. We first describe the spatial structure of 2DWC. Then the exchange interaction of electron spins at the lower coordinate energy level will be described. After this, the Ising model will be formulated, which determines the spin statistics, spin structure and susceptibility. Next, we present the results of modeling the spin structure of circular clusters, as well as the dependence of the spin magnetic susceptibility on the magnetic field. After this, we calculate the spin correlation functions at low temperature and zero magnetic field. Finally, we discuss the results obtained. The applicability of the Ising model for describing the system of heavy holes in a two-dimensional semiconductor quantum well is discussed in the Appendix.

## 2. Clusters under study

Let us consider free electrons near a flat surface, placed in a potential well $k(x^2 + b^2y^2)/2$, $(k > 0)$, and experiencing Coulomb repulsion $\sum\limits_{i>j} \frac{e^2}{\varepsilon|\mathbf{r}_i - \mathbf{r}_j|}$, where $e$ — electron charge, $\varepsilon$ — dielectric constant of the environment, $b$ — anisotropy parameter of the potential well. Next we will use dimensionless coordinates, scaling

$$\mathbf{r}_i \to L\mathbf{r}_i, \quad L = (2e^2/\varepsilon k)^{1/3}.$$

Energy measured in units

$$E_0 = (ke^4/2\varepsilon^2)^{1/3}$$

is as follows:

$$H = \sum_{i>j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} + \sum_i (x_i^2 + b^2 y_i^2). \qquad (1)$$

The asymmetry of the wells determines the corresponding asymmetry of the two-dimensional cluster. At $b = 1$, clusters on average have circular symmetry; at $b \gg 1$ or $b \ll 1$ the cluster becomes elongated and, in the limit, one-dimensional. At low temperatures electrons occupy energy minima, forming a quasiperiodic lattice that minimizes the cluster energy.

Estimating the size of 2DwC with $n$electrons gives $R \sim (ne^2/\varepsilon k)^{1/3}$, the distance between electrons is $R/\sqrt{n}$ and the characteristic electron density $n/R^2$ [26].

To find the spin structure, it is necessary to take into account the quantization of electrons. The wave functions of one electron, when the others are fixed, are localized near the bottom of wells and have energies $\varepsilon_0 = \hbar(k/m_e)^{1/2}n^{1/4}$, where $m_e$ — electron mass.

The spin ordering is determined by the spin Hamiltonian, which will be studied on the basis of the electron coordinates found from the Hamiltonian (1). We neglect the influence of spin interaction on the cluster structure. Besides, we assume that all electrons are in the ground coordinate state. The wave function of the ground state of $i$-th electron decreases as

$$\psi(\mathbf{r} - \mathbf{r}_i) \propto \exp(-\alpha|\mathbf{r}_i - \mathbf{r}|),$$

where $\alpha \sim \sqrt{m_e\varepsilon_0}L/\hbar$. The exchange interaction between the spins of two electrons in the ground state is determined

by the integral of overlap between the coordinate wave functions $J_{ij} = J_0 \exp(-\alpha|\mathbf{r}_i - \mathbf{r}_j|)$ and is antiferromagnetic in nature. Here and below we neglect the difference in states of different electrons.

Note that exponential behavior is not universal and does not take into account either the lattice or the magnetic field. In the presence of the magnetic field, the long-range dependence is replaced by $\propto \exp(-r_{ij}^2/a_B^2)$, where $a_B = \sqrt{c\hbar/eB}$ is the magnetic length, $B$ — magnetic field, $c$ — speed of light. This dependence leads to exchange interaction controlled by the magnetic field. But at $r_{ij}/a_B^2 \ll \alpha$ the magnetic field is insignificant for $J_{ij}$.

We will use the Ising model to describe spin ordering. This is a widely used model in the theory of phase transitions. The applicability of the Ising model can be confirmed for hole systems in conventional semiconductors (see Appendix). In magnetic field $B$ the spin energy $E$ has form

$$E = \sum J_{ij}\sigma_i\sigma_j - h\sum \sigma_i, \quad \sigma_j = \pm 1, \tag{2}$$

where $h = 2\mu_b B/E_0$ and $\mu_B$ — Bohr magneton. Statistical sum

$$Z = \sum_{(\sigma_i=\pm 1)} \exp(-\beta E). \tag{3}$$

Here $\beta = E_0/k_B T$, $T$ — temperature, $k_B$ — Boltzmann constant. Variable $Z$ determines all thermodynamic properties of the system, in particular, induced magnetic moment $M$ and spin magnetic susceptibility $\chi$:

$$M = \frac{1}{\beta}\frac{\partial \ln Z}{\partial h}, \quad \chi = \frac{\partial M}{\partial h}. \tag{4}$$

We performed numerical calculations based on the equations (2–4).

## 3. Magnetic moment

The results of the numerical calculation of the magnetic moment dependence on the magnetic field are presented in Figure 1. The magnetic moment of 2DWC increases with the magnetic field to a maximum value equal to $n$. The calculation shows that the dependence of the magnetic moment on $h$ differs significantly from the Curie–Weiss model

$$M(h) = M_0 \tanh(\gamma\beta h).$$

At high temperatures (black dashed lines), the growth is more linear than would be expected from the Curie–Weiss model (red dashed lines). We see that $M(h) \propto h$ for small $h$. Then this dependence abruptly reaches $M(h) = M_0$. Another important property of $M(h)$ is the presence of steps. The steps are explained by the sequential spin pairs depairing by the magnetic field. This result is in agreement with the periodic peak structure $\chi(h)$.

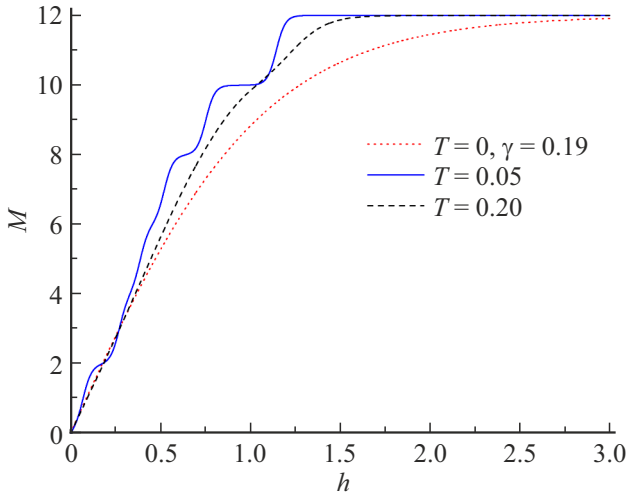


**Figure 1.** Spin magnetic moment of round cluster with $n = 12$ electrons vs. magnetic field at relatively high temperature $T = 0.2$ (black dashed line) and low temperature $T = 0.05$ (blue, solid line). (See color illustration online.)

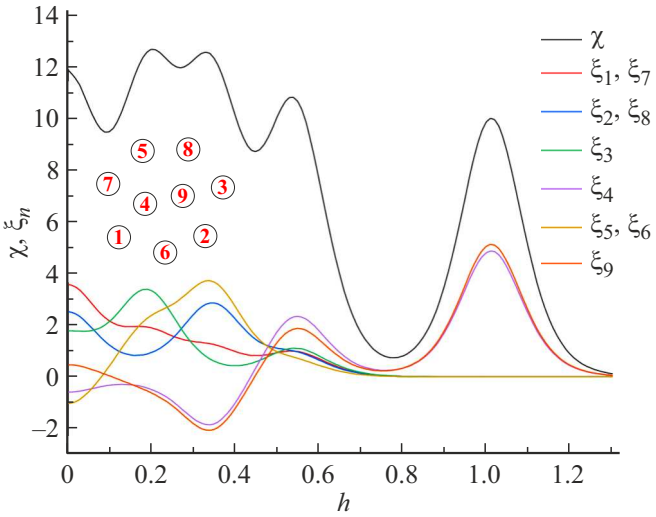


**Figure 2.** Spin magnetic susceptibility $\chi(h)$ of round 2DWC with 9 electrons and derivative of average spin of $k$-th electron $\xi_k(h)$ depending on the magnetic field at temperature $T = 0.1$. Structure of round 2DWC with 9 electrons. Digits inside circles enumerate electrons. The peaks $\chi(h)$ can be attributed to specific electrons. (See color illustration online.)

## 4. Spin magnetic susceptibility

The spin magnetic susceptibility $\chi(h)$of round 2DWCs is demonstrated in Figure 2. The dependence contains peaks arising as a result of sequential spins depairing.

Round 2DWCs have a shell structure (Figure 3). Electrons in different shells have very different bonds with other shells. In the cases of 2–5 electrons they all form a single shell with symmetry towards the replacement $i \to j$.

Starting from $n = 6$ two or more shells appear. The only electron in the center at $n = 6$ or $n = 7$ remains unpaired.

Round 2DWCs (see Figure 2) have a complex spin ordering. If the distance between the shells exceeds the interelectron distance inside the shell, then the spin ordering between the inner and outer shells occurs at much less $h$, than the ordering in the shell. However, if the hell contains odd number of electrons, a the depaired electron in one shell can pair with electron in another shell, but with a lower pairing energy. so, the pairing of distant electrons can be superimposed on the simple single shell pattern, creating more complex peak structure $\chi(h)$. We see this behavior in Figure 2.

Note that more complex pairing options are possible. In particular, direct exchange of two distant spins is very weak. In this case, indirect pairing predominates [27]. The example of such a situation is presented in Figure 2, where electrons experience the transition from antiferromagnetic ordering at $h < J_{ij}$ to ferromagnetic ordering at $h > J_{ij}$.

## 5. Spin correlation function

The spin structure of the Wigner cluster can be effectively identified using the spin correlation function $\xi_{i,j} = \langle \sigma_i \sigma_j \rangle$, where $\langle \dots \rangle$ denotes statistical averaging. Unlike the infinite 2DWL, $\xi_{ij}$ depends on the coordinates of two spins. The susceptibility of 2DWL $\chi$ smoothly depends on $h$. On the contrary, in 2DWC, upon change in $h$, multiple rearrangements occur, which leads to a periodic dependence of susceptibility $\chi$ on $h$.

A common feature of different clusters is the pairing of the most tightly bound nearby electrons, if there is an even number of them. They comprise „molecules“. Electrons that do not have a pair in this close community are paired with more distant ones. If there are several such electrons, then they have an unpaired spin. However, in the system with many electrons different types of pairing are mixed.

Figure 3 shows the variables $\xi_{i,j}$ for $i$ chosen to have the smallest value of $r_i$. We set the zero temperature limit to highlight the antiferromagnetic ordering of spins. The correlation function in round clusters decreases with distance $|\mathbf{r}_i - \mathbf{r}_j|$. The drop follows from statistical considerations. The imperfection of the crystallographic lattice also contributes to the attenuation of the correlation function.

Below we analyze the dependence $\xi_{i,j}$ for round clusters with small number of electrons. The weaker interaction between the shells and the smaller distance between the inner electrons make their spins more coupled. The remaining spins adapt to the spins of the inner shells. Systems with a small number of electrons $n = 3, 4, 5$ form regular polygons. In these polygons the situation depends on the parity $n$.

Let us analyze the resulting correlation functions. At zero $h$ the systems with even number of electrons tend to pair all electrons, while in system with odd electrons one electron remains unpaired.

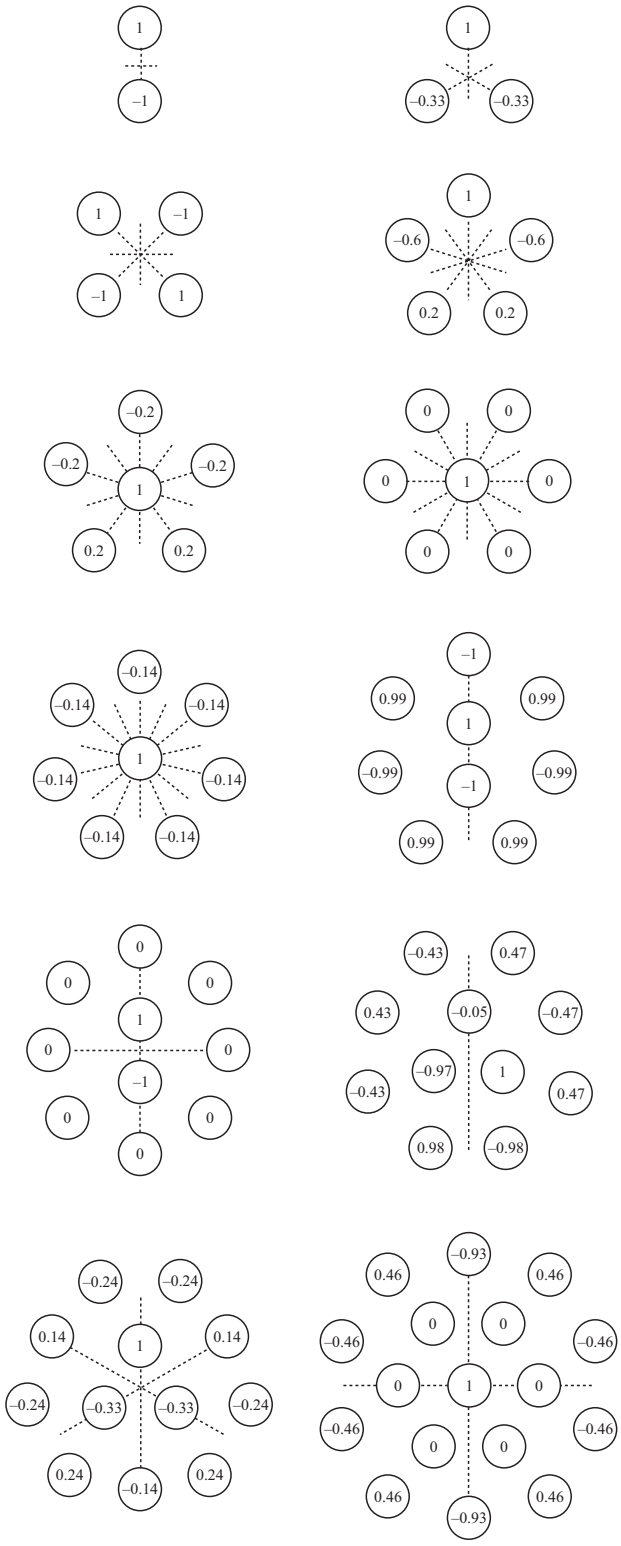


**Figure 3.** Correlation function $\xi_{ij} = \langle \sigma_i \sigma_j \rangle$ of spins for 2DWC with $n = 2 \dots 12$ and $n = 17$. The centers of the circles correspond to the positions of the electrons. As number $i = 1$ we selected electron with the lowest $r_i$. The value $\xi_{1,j}$ is shown inside circle representing the corresponding electron located at the corresponding location. The limit $h \to 0$ $\beta \to \infty$ is represented. The lines of exact and approximate reflection planes are indicated by dashed lines and dots, respectively. Antiferromagnetic ordering is visible.

In a system with 2, 3, 4 and 5 electrons, the particles are located equidistantly. All positions are equal. You can easily count all configurations with the same lowest energy and find the correlation functions (see Figure 3).

2DWCs with 2 and 4 electrons are completely polarized, and all $\xi_{1,j\neq i} = \pm 1$. Let us consider the case $n = 3$. Let $i = 1$, and $\sigma_1 = 1$ fixed. Minimum energy is achieved for configurations $(1, 1, -1)$, $(1, -1, -1)$, $(1, -1, 1)$. Averaging product $\sigma_1\sigma_j$, we obtain $\xi_{1,1} = 1$, $\xi_{1,2} = \xi_{1,3} = -1/3$. Similarly for $n = 5$ we obtain $\xi_{1,1} = 1$, $\xi_{1,2} = \xi_{1,5} = -3/5$, $\xi_{1,3} = \xi_{1,4} = 1/5$.

At $n = 6$ one electron is in the center, and the rest occupy the vertices of a regular pentagon. The central spin is connected to one spin on the outer shell. When distributed over 5 sites, one spin gives $\xi_{1,j\neq 1} = -1/5$.

A cluster with $n = 7$ contains 1 electron in the inner shell and 6 paired electrons on the outer shell. This gives $\xi_{1,j\neq 1} = 0$.

The case $n = 8$ is similar to the case $n = 6$: one common spin in seven sites of the outer shell gives $\xi_{1,j\neq 1} = -1/7$.

At $n = 9$ and $n = 10$ there is no axial symmetry of the spatial structure, but there is a reflection plane. The two central, closely spaced spins are strongly coupled, and $\xi_{1,j\neq 1} = \pm 1$. The outer electrons have at $n = 9$ $\xi_{1,j\neq 1} \approx \pm 1$, keeping alternation to minimize total energy, except for two spins for which alternation is not possible. At $n = 10$ there are 8 paired electrons on the outer shell, which gives $\xi_{1,j} = 0$. The appearance of one more electron made possible the spins alternation in the outer shell.

Systems with $n = 11$ and $n = 12$ also have central almost regular triangles. However, the order of the spins in these cases is very different. This can be explained by the symmetry of the systems. In case of $n = 11$ there is a plane of reflection, and in the case of $n = 12$ there is symmetry $C_{3v}$.

At $n = 11$, the spatial structure of 2DWC has a reflection plane, and the correlation function $\xi_{1,j\neq 1}$ is antisymmetric with respect to it. As a result, one of the variables $\xi_{1,j\neq 1}$ in the inner shell is small, and the other has a value about $-1$. The four electrons on the top right, like 2DWC with $n = 4$ have $\xi_{1,j} \approx \pm 1$. The remaining spins have alternating values of order $\pm 0.5$.

The inner shell of a 12 electron cluster has spin ordering as system with 3 electrons. The spins of the outer shell adjust to them.

Now let's analyze the cluster with $n = 17$. This cluster contains shells with 1, 6 and 10 electrons. The second and third shells form approximately regular hexagon and decagon. These shapes have mutual elements of symmetry: two orthogonal planes of reflection and rotation by an angle $\pi$, thereby conforming with group $C_{2v}$. An even number of electrons in the second shell leads to their complete pairing with $\xi_{i,j}$ for $j$ within this group. Symmetry leads to the electrons separation on the outer shell into 3 subsets with 2, 4 and 4 electrons, which have the same values of $\xi_{1,j}$. The subset with $\xi_{1,j} = -0.93$ is closer to the origin (and interacts more strongly with the central electron), while the others are paired with the nearest electrons from this shell and have alternating spins.

Thus, the qualitative behavior of the correlation function becomes completely understandable.

## 6. Relation between rearrangement of spin structure and spin susceptibility

The presence of several peaks allows them to be assigned to specific spin pairs. This can be called „spin pairing spectroscopy". The maximum value of $h$ corresponds to the bond strength. In the simple case, this value coincides with $J_{ij}$ for specific pair $\{i, j\}$. However, if the distance between electrons $r_{i,j}$ is large, the exponential decay $J_{ij}$ makes direct exchange too weak. In this case, the indirect effect along a certain chain of reorienting spins coupling $i$ with $j$ can lead to stronger power-law interaction between $\sigma_i$ and $\sigma_j$. Here we analyze different clusters in different fields to systematize the types of spin ordering.

For example, consider the round cluster with $n = 9$ electrons, two of which are located within the cluster at small distance, and the rest make up the outer shell with approximately the same distances between them. The corresponding dependence $\chi(h)$ is shown in Figure 2. To match the peaks we also prepare the derivative with respect to $h$ of the average spin of each individual electron. We see that all peaks $\chi(h)$ can be attributed to certain spins and spin pairs. The inner electrons are located close to each other. Then the peak at $h = 0$ is associated with unpaired electron located on the outer shell. This electron is more easily oriented by the weak $h$. This leads to maxima of the average spin derivatives $\xi_n(h) = d\langle\sigma_n(h)\rangle/dh$ for shell electrons with numbers $n = 1$, 2, 7, 8, 9. In the region of large $h$ there is only one peak at $h \approx 1$, caused by the strongest bound pair of the 4th and 9th electrons. The peak at $h = 0.2$ can be attributed to 3rd electron (and partially to the minima for the 2nd and 8th electron). The peak at $h = 0.35$ is mainly associated with the peaks for spins of the 5th, 6th, 2nd and 8th electrons and minima for the 4th and 9th electrons.

Thus, the origin of the peaks can be understood from the geometric structure of the cluster, and the magnetic susceptibility spectrum allows one to study the spatial organization of the clusters.

Note that, as can be seen in Figure 2, pairing-depairing does not necessarily occur with pair of nearest electrons. This is also reflected in the spins of other electrons (see the maximum at $h = 0.2$ for $\xi_3$, which is accompanied by minima for $\xi_2$ and $\xi_8$ and a shoulder on the curve for $\xi_5$ and $\xi_6$).

## 7. Estimates of characteristic parameters

We expect that the most clear manifestation of 2DWC can be obtained in systems with a relatively low dielectric constant and a large electron mass. Besides, the purity of

the material is important to avoid potential fluctuations that localize holes. To get closer to the experimental situation, we estimated the parameters characterizing 2DWC for various systems where, we hope, electronic spatial and spin ordering are significant. The most suitable systems are electrons on the surface of liquid helium or freely suspended layers of semiconductor of *p*-type. To eliminate the Fermi energy, the charge density is chosen to be quite low. However, this reduces the maximum melting temperature of 2DWC. We cannot lower the electron concentration much because the exchange interaction falls exponentially with the distance between the electrons. Let us consider electrons on He surface, a *p*-channel of GaAs type, and a two-dimensional material $MoS_2$. It is convenient to choose exchange integrals by several times less than unity.

Let us estimate the characteristic parameters. Let $n = 12$ (Figure 1), $k = 1.66 \cdot 10^{-2}$ meV/cm$^2$, $\varepsilon = 1$ (free suspended layer). Then $R = 6.18 \cdot 10^{-3}$ cm, $n_s = 10^5$ cm$^{-2}$, unit of energy and temperature is $E_0 = 0.26$ meV$= 2.99$ K, and unit of length is $L = 5.58 \cdot 10^{-4}$ cm. Dimensionless temperatures on the graphs 0.05 and 0.2 correspond to $T = 0.15$ K and $T = 0.6$ K. The corresponding maximum exchange energies $J_{ij}$ are 0.043 meV (He), 0.11 meV (GaAs), 0.093 meV ($MoS_2$). Characteristic magnetic fields for $h = 1$ are $B = 2.23$ T (He), $B = 1$ T (GaAs), $B = 1.43$T ($MoS_2$). These values seem to quite extreme, but achievable.

## 8. Conclusion and discussion

We studied the spin magnetic moment, correlation function, and magnetic susceptibility of round two-dimensional Wigner clusters. It was found that the dependence of the magnetic moment on the magnetic field differs from Curie–Weiss law and is practically linear up to the saturation value at relatively high temperatures. Low-temperature magnetic susceptibility at low magnetic fields is zero for an even number of electrons and finite for an odd number of electrons, which is explained by spin pairing.

Magnetic susceptibility fluctuates at finite magnetic field due to successive pairing from the weakest to the strongest spin bonds. These oscillations show weak regularity, which is associated with the shell structure of the cluster (weak bond between electrons of different shells), the approximate periodicity of the electrons arrangement on the shell, and the cyclicity of individual shell (absence of ends). As a result of the shell structure, the different shells act almost like independent subsystems. The periodicity of the electrons arrangement in the shell makes all electrons equally susceptible to the magnetic field, but intershell pairing leads to uneven magnetic oscillations.

Let us compare the correlation function of the 2DWC with the infinite lattice of the 2DWL. The infinite 2DWL is homogeneous. In the Ising lattice at $h = 0$ the phase transition occurs at a certain temperature $T_c$. Near $T_c$ the correlation radius grows infinitely, and the correlation function becomes the power function. This behavior determines the similarity laws for susceptibility at $\tau = (T_c - T)/T_c \to 0$ and $h \to 0$. It is obvious that this scaling behavior will be preserved to some extent in the inhomogeneous system such as 2DWC, when the correlation length becomes smaller than the cluster radius.

In the infinite system the 2nd kind phase transition is accompanied by the infinite correlation length. The two-dimensional triangular lattice undergoes a single phase transition. In a finite magnetic field no phase transition occurs. The susceptibility $\chi$ in a magnetic field smoothly depends on $h$. On the contrary, in 2DWC, upon change in $h$, multiple rearrangements occur, which leads to a periodic dependence of susceptibility $\chi$ on $h$.

Note that the action of the magnetic field can be replaced by cluster rotation with angular frequency $\Omega = \frac{eB}{mc}$. 2DWC in an axially symmetric potential is capable of free rotation without changing shape and destruction [26], which makes it possible to use rotation instead of the magnetic field.

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### Conflict of interest

The authors declare that they have no conflict of interest.

## Appendix. Applicability of Ising model

Ising model is based on replacing the quantum spin operator $\hat{s_i}$ of spin 1/2 with the classical *c*-number $\sigma_i$. This reduces $2^n$-dimensional spin Hilbert state to a space of *n*-dimensional binary numbers $\sigma_i$, reducing an NP-complete problem to NP-incomplete one. From a physical point of view this can be justified when, instead of the Pauli matrices $2 \times 2$ the Hamiltonian will include only one of them, for example, $\sigma = \hat{s}_z$. This is the case when the spin Hamiltonian has the form

$$\sum_{ij} J_{ij,\alpha\beta} \hat{s}_{i\alpha} \hat{s}_{j\beta},$$

where $\alpha$ and $\beta$ denote Cartesian coordinates $(x, y, z)$, and $J_{ij,zz} \gg J_{ij,xx}$, $J_{ij,zz} \ll J_{ij,yy}$. If this is so, then only diagonal components of the spin, $\pm 1$, are taken into account, and the Heisenberg Hamiltonian is transformed into the Ising Hamiltonian.

However, symmetry allows such a case in simple models $J_{ij,\alpha,\beta} = J_{ij}\delta_{\alpha\beta}$. In this case, it is impossible to transform the Heisenberg model into the Ising model. Then Ising's approach is only an approximation to the more general Heisenberg model.

However, there are systems in which this is not the case. Imagine hole Hamiltonian in a regular cubic semiconductor

such as GaAs. In the volumetric isotropic case, this Hamiltonian has the following form:

$$H = Ap^2 + B(\mathbf{pj})^2,$$

where $\mathbf{j}$ — operator of spin 3/2. Surface quantization splits the states $m = \pm 3/2$ and $m = \pm 1/2$ so that 3/2 usually becomes the top state of the heavy hole. The doubly degenerate states $m = \pm 3/2$ are similar to the state $\pm 1/2$ of the free electron, excluding that now the exchange integral is $J_{ij,\alpha\beta} \approx J_{ij}\delta_{\alpha z}\delta_{\beta z}$. In fact, the Hamiltonian of the exchange interaction has the following form:

$$H_{ex} \propto (\mathbf{j}_{i,\alpha}\mathbf{j}_{j,\beta})$$

Projecting $\mathbf{j}_{i,\alpha}$, $\mathbf{j}_{j,\beta}$ on subspace of states with $j_z = \{3/2, -3/2\}$ we see that $\langle \pm 3/2|j_x| \pm 3/2\rangle$, $\langle \pm 3/2|j_y| \pm 3/2\rangle = 0$, $\langle 3/2|j_z|3/2\rangle = 3/2$. This diagonalizes the Hamiltonian and transforms it into the Ising Hamiltonian. On the contrary, for states with $j_z = \pm 1/2$, $\langle \pm 1/2|j_x| \pm 1/2\rangle \neq 0$, $\langle \pm 1/2|j_y| \pm 1/2\rangle \neq 0$, this situation shall be described by the Heisenberg model.

Therefore, such holes are good candidates for creating Wigner two-dimensional hole clusters controlled by the Ising Hamiltonian.

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