Solution of the equations system of nonlinear electrodynamics minimally coupled to gravity in the axisymmetric case

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The main features of electromagnetic fields follow from the analysis of regular solutions of the system of equations of nonlinear electrodynamic minimally coupled to gravity (NED-GR). In this paper, a representation for the derivative of the Lagrangian with respect to the field invariant, which ensures the consistency of the system of equations NED-GR, is found and the exact solution of this system of equations is obtained. Expressions for the components of the electromagnetic field are given.

Keywords: nonlinear electrodynamics, system of equations, consistency condition, electromagnetic field components.

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Introduction

Electrically charged objects coupled by electromagnetic and gravitational interactions are described in the general formulation by nonlinear electrodynamics related to gravity (NED-GR) Nonlinear electrodynamics (NED) was proposed by Born and Infeld in 1934 in order to consider the electromagnetic field and particles within the same physical system and to provide finite values of physical quantities [1]. These goals are achieved in NED-GR, which allows for regular solutions describing compact finite energy objects coupled by electromagnetic and gravitational interactions [2]. A detailed review of the literature is given in [2]. In the paper [3] it was found that the NED-GR system of equations is inconsistent in the case of arbitrary Lagrangian.

In this paper, firstly, an analytical expression has been found for the parametric function $L_F(r, \theta)$ (the derivative of the Lagrangian with respect to the field invariant) for which the system of equations NED-GR is consistent, and secondly, the exact solutions of this system for the found function have been obtained.

1. A necessary and sufficient condition for the consistency of a system of equations

The system of equations NED-GR for the two components of the electromagnetic field $F_{10}(r, \theta)$, $F_{20}(r, \theta)$ in the axially symmetric case has the form [3]:

$$\frac{\partial}{\partial r}\,[(r^2 + a^2)\sin(\theta) L_F F_{10}] + \frac{\partial}{\partial \theta}\,[\sin(\theta) L_F F_{20}] = 0, \quad (1)$$

$$\frac{\partial}{\partial r}\,[a \sin(\theta) L_F F_{10}] + \frac{\partial}{\partial \theta}\,[\frac{1}{a\sin(\theta)} L_F F_{20}] = 0, \quad (2)$$

$$\frac{\partial F_{20}}{\partial r} - \frac{\partial F_{10}}{\partial \theta} = 0,$$

$$\frac{\partial}{\partial \theta}\,[a^2 \sin^2(\theta) F_{10}] - \frac{\partial}{\partial r}\,[(r^2 + a^2) F_{20}] = 0, \quad (3)$$

where a — is the angular pulse (Kerr parameter), (r, θ) — Boyer-Lindquist coordinates (Cartesian coordinates x, y, z are connected with Boyer-Lindquist coordinate by the relations $x^2+y^2=(r^2+a^2)\sin^2(\theta)$; $z = r\cos(\theta)$). Here $L_F(r, \theta)$ is the sought function, which plays the role of a parameter. The equation that the function $L_F(r, \theta)$ must satisfy in order to provide a necessary and sufficient condition for the consistency of the system of equations (1)−(3) has the form [3]:

$$\frac{\partial}{\partial r}\left(\frac{1}{L_F}\frac{\partial L_F}{\partial \theta}\right)\frac{\partial}{\partial \theta}\left(\frac{1}{L_F}\frac{\partial L_F}{\partial r}\right) + \frac{4a^2\sin^2(\theta)}{\Sigma^2}\frac{1}{L_F^2} \times \left[r\frac{\partial L_F}{\partial r} + \cot(\theta)\frac{\partial L_F}{\partial \theta}\right]^2 = 0. \quad (4)$$

Here $\Sigma = r^2 + a^2\cos^2(\theta)$. If the function $L_F(r, \theta)$ is twice continuously differentiable, then the sum of squares must be zero, therefore:

$$\frac{\partial}{\partial r}\left(\frac{1}{L_F}\frac{\partial L_F}{\partial \theta}\right) = 0$$

and

$$r\frac{\partial L_F}{\partial r} + \cot(\theta)\frac{\partial L_F}{\partial \theta} = 0$$

simultaneously. Solving the system of these equations, we find the general solution of the equation (4)

$$L_F(r, \theta) = C_0\big(r\cos(\theta)\big)^{\mu}, \quad (5)$$

where C_0, μ — arbitrary real numbers. Thus, the system of equations (1)−(3) is consistent if and only if the parametric function L_F has the form (5). In the future, we work only with the joint system.

2. Finding the Exact System Solution

Let us introduce the notation $U_{10} = L_F F_{10}$, $U_{20} = L_F F_{20}$. Taking into account these notations, the system of equations (1)–(2) can be reduced to the following form:

$$U_{10} = \frac{1}{2a^2 r \sin(\theta)} \frac{\partial}{\partial\theta}\left(\frac{\Sigma}{\sin(\theta)} U_{20}\right),$$

$$\frac{\partial}{\partial\theta}\left\{\frac{1}{\sin(\theta)}\left[\frac{\partial}{\partial r}\left(\frac{\Sigma}{r} U_{20}\right) + 2U_{20}\right]\right\} = 0. \quad (6)$$

The general solution of the second equation (6) can be written in the form

$$U_{20}(r,\theta) = \frac{r}{\Sigma^2}\left[\Psi(\theta) + \sin(\theta)\int \Phi(r)\Sigma(r,\theta)dr\right], \quad (7)$$

where $\Phi(r)$ — arbitrary function r, $\Psi(\theta)$ — arbitrary function θ. A system of equations (3) can be reduced to the form

$$F_{10} = \frac{1}{a^2 \sin(2\theta)} \frac{\partial}{\partial r}(\Sigma F_{20}),$$

$$\frac{\partial}{\partial r}\left\{\frac{\partial}{\partial\theta}\left(\frac{\Sigma}{\sin(2\theta)} F_{20}\right) - a^2 F_{20}\right\} = 0. \quad (8)$$

Let us move to functions U_{10}, U_{20} in equations (8) and will assume that the function L_F is of the form (5), then the first of the equations (8) can be written as

$$U_{10} = \frac{1}{a^2\sin(2\theta)}\left[\frac{\partial}{\partial r}\left(\Sigma U_{20}\right) - \mu\frac{\Sigma}{r}U_{20}\right].$$

On the other hand, the function U_{10} satisfies the first of equations (6). By equating these two expressions, we obtain the first additional equation that must be satisfied by the function U_{20}, which is the solution of the system (1)–(2):

$$r\frac{\partial}{\partial r}\left(\Sigma U_{20}\right) - \cos(\theta)\frac{\partial}{\partial\theta}\left(\frac{\Sigma}{\sin(\theta)}U_{20}\right) = \mu\Sigma U_{20}. \quad (9)$$

The second of the equations (8), after moving to the function U_{20} and take into account (5), gives a second additional equation for finding the function $U_{20}(r,\theta)$. After simplification this equation, will be written as

$$\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left(\Sigma U_{20}\right) + (r^2 - a^2\cos^2(\theta) - \mu\Sigma)U_{20}\right\} - \mu\frac{r^2 - a^2\cos^2(\theta)}{r}U_{20} = 0. \quad (10)$$

Thus, there are two additional equations (9) and (10) that the function U_{20} must satisfy. This function, as the general solution of the second equation (6), depends on two arbitrary functions $\Phi(r)$, $\Psi(\theta)$ (expression (7)). By substituting the representation (7) into the second additional equation (10), we obtain (in assumption $\sin(\theta) \neq 0$) a linear ordinary differential equation of the first order to find the function $\Phi(r)$ of the following form:

$$\frac{d}{dr}\left(r^2\Phi(r)\right) - \mu r\,\Phi(r) = 0.$$

Its general solution will be as follows

$$\Phi(r) = D_1\, r^{\mu-2},$$

where D_1 — arbitrary constant. By substituting (7) into the first additional equation (9), we obtain an integro-differential equation with respect to the functions $\Phi(r)$ and $\Psi(\theta)$ with the parameter μ as follows:

$$\Psi'(\theta) + [(1+\mu)\tan(\theta) - \cot(\theta)]\Psi(\theta) = \sin(\theta)\tan(\theta) \times\left[r^3\Phi(r) - (1+\mu)\int\Phi(r)r^2dr\right] + a^2\sin^2(\theta)\cos(\theta) \times\left[(1-\mu)\int\Phi(r)dr + r\Phi(r)\right]. \quad (11)$$

Let us substitute the found function $\Phi(r)$ into the right-hand side of the equation (11). Let $\mu \neq \pm 1$, then

$$\int\Phi(r)dr = D_1\frac{r^{\mu-1}}{\mu-1}, \quad \int\Phi(r)r^2dr = D_1\frac{r^{\mu+1}}{\mu+1},$$

the right-hand side of the equation (11) will turn to zero, and equation (11) will take the form

$$\Psi'(\theta) + [(1+\mu)\tan(\theta) - \cot(\theta)]\,\Psi(\theta) = 0. \quad (12)$$

The general solution of this equation is

$$\Psi(\theta) = D_2\sin(2\theta)\big(\cos(\theta)\big)^{\mu}, \quad (13)$$

where D_2 — arbitrary constant. Thus, in the case of $\mu \neq \pm 1$ the formula (13) yields $\Psi(\theta)$ and $\Phi(r) = D_1 r^{\mu-2}$.

Let $\mu = +1$, then

$$\Phi(r) = D_1\, r^{-1},$$

$$\int\Phi(r)dr = D_1\ln(r), \quad \int\Phi(r)r^2dr = 0.5D_1r^2,$$

and the equation (11) looks like as follows:

$$\Psi'(\theta) + [2\tan(\theta) - \cot(\theta)]\,\Psi(\theta) = D_1a^2\sin^2(\theta)\cos(\theta).$$

Its general solution is

$$\Psi(\theta) = \sin(2\theta)\cos(\theta)\big\{D_2 - 0.5\,D_1a^2\ln|\cos(\theta)|\big\}. \quad (14)$$

Thus, in the case of $\mu = +1$, the formula (14) yields $\Psi(\theta)$, and $\Phi(r) = D_1r^{-1}$.

Let $\mu = -1$, then

$$\Phi(r) = D_1\, r^{-3},$$

$$\int\Phi(r)dr = -0.5\,D_1(r)^{-2}, \quad \int\Phi(r)r^2dr = D_1\ln(r),$$

and the equation (11) looks like as follows:

$$\Psi'(\theta) - \cot(\theta)\Psi(\theta) = D_1\sin(\theta)\tan(\theta).$$

Its general solution is

$$\Psi(\theta) = \sin(\theta)\{2D_2 - D_1 \ln|\cos(\theta)|\}. \quad (15)$$

Thus, in the case of $\mu = -1$ we have $\Phi(r) = D_1 r^{-3}$, and $\Psi(\theta)$ gives the expression (15).

Once the functions $\Phi(r)$ and $\Psi(\theta)$ have been found, we can find the function U_{20} by formula (7), U_{10} — by the first of formulas (6), and then find the components of the electromagnetic field $F_{10}(r, \theta)$, $F_{20}(r, \theta)$.

As a result, we obtain:

field components in the case of $\mu \neq \pm 1$:

$$F_{10}(r, \theta) = -D_1 \frac{2r\cos(\theta)}{(\mu^2 - 1)\Sigma^2} \left(\cos(\theta)\right)^{-\mu} - D_2 \frac{r^2 - a^2\cos^2(\theta) + \mu\Sigma}{a^2\Sigma^2} (r)^{-\mu},$$

$$F_{20}(r, \theta) = -D_1 \frac{\sin(\theta)}{(\mu^2 - 1)\Sigma^2} \left\{r^2 - a^2\cos^2(\theta) - \mu\Sigma\right\} \times \left(\cos(\theta)\right)^{-\mu} + D_2 \frac{r\sin(2\theta)}{\Sigma^2} (r)^{-\mu}.$$

In particular, for the case $\mu = 0$, we will have, in contrast to the result given in [4]:

$$F_{10} = D_1 \frac{2r\cos(\theta)}{\Sigma^2} - D_2 \frac{r^2 - a^2\cos^2(\theta)}{a^2\Sigma^2},$$

$$F_{20} = D_1 \frac{\sin(\theta)(r^2 - a^2\cos^2(\theta))}{\Sigma^2} + D_2 \frac{r\sin(2\theta)}{\Sigma^2}.$$

For $D_1 = 0$, $D_2 = -qa^2$ we get the known solution [4].

In the case of $\mu = +1$:

$$F_{10}(r, \theta) = D_1 \frac{1}{2r\Sigma^2}\left\{2r^2 + a^2\cos^2(\theta) - 2r^2 \ln\left(\frac{r}{|\cos(\theta)|}\right)\right\} - D_2 \frac{2r}{a^2\Sigma^2},$$

$$F_{20}(r, \theta) = D_1 \frac{\tan(\theta)}{\Sigma^2}\left\{0.5r^2 + a^2\cos^2(\theta)\ln\left(\frac{r}{|\cos(\theta)|}\right)\right\} + D_2 \frac{\sin(2\theta)}{\Sigma^2}.$$

And finally, in the case of $\mu = -1$:

$$F_{10}(r, \theta) = D_1 \frac{r}{2a^2\Sigma^2}\left\{r^2 + 2a^2\cos^2(\theta) + 2a^2\cos^2(\theta) \times \ln\left(\frac{r}{|\cos(\theta)|}\right)\right\} + D_2 \frac{2r\cos^2(\theta)}{\Sigma^2},$$

$$F_{20}(r, \theta) = D_1 \frac{0.5r^2\sin(2\theta)}{\Sigma^2}\left\{-0.5a^2\cos^2(\theta)r^{-2} + \ln\left(\frac{r}{|\cos(\theta)|}\right)\right\} + D_2 \frac{r^2\sin(2\theta)}{\Sigma^2}.$$

Conclusion

In this paper, an analytical expression for the derivative of the Lagrangian with respect to the field invariant, which ensures the fulfillment of the necessary and sufficient condition for the consistency of the NED-GR system of equations is obtained. For the found function $L_F(r, \theta)$, an exact solution of this system of equations is received.

Analytical expressions for the components of the electromagnetic field are given, which open up opportunities for studying their behavior at different values of the parameter μ, which defines the form of the parametric function $L_F(r, \theta)$. Of undoubted interest is the study of the asymptotic behavior of the obtained solutions at $r \to 0$ and $r \to \infty$, for example, the asymptotic behavior at $r \to 0$ will reveal the features of electromagnetic dynamics on the de Sitter vacuum disk. A more detailed analysis of these issues will be the subject of further research.

Conflict of interest

The author declares that he has no conflict of interest.

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