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Dynamics of thermal pairwise entanglement of qubits in the three-qubit Tavis–Cummings model

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An exact solution has been found for a model consisting of three identical qubits, one of which is in a free state, and the other two are trapped in an ideal cavity and resonantly interact with the selected mode of this cavity. Based on the exact solution, the pairwise negativities of qubits was calculated for two initial W-type qubits states and the thermal state of the cavity field. The influence of the intensity of the thermal noise of the cavity and the parameters that specify the initial state of the qubits on the amount of their entanglement in the process of further evolution has been studied. It is shown that in the case of low intensities of the thermal field of the cavity, for one of the initial states of the qubits under consideration, the phenomenon of sudden death of entanglement is observed, while for the other initial state of the qubits such an phenomenon is absent. It has also been established that with an increase in the intensity of the thermal field, the sudden death of entanglement occurs for both states.

Keywords: qubits, genuine entangled W-type states, thermal field, one-photon transitions, entanglement, negativity, sudden death of entanglement.

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Introduction

Multi-qubit quantum entangled states are a fundamental resource in the physics of quantum computing, quantum communications, quantum cryptography, quantum metrology, etc. [1–6]. The difficulties of experimental and theoretical description of entangled states increase significantly with increasing number of qubits in the system. Therefore, at present, special attention is paid to the theoretical and experimental study of three-qubit systems (see links in [7]). The importance of studying such systems is also due to the fact that they can be used to create universal three-qubit gates, which are an alternative to the universal two-qubit gate of controlled negation and one-qubit rotations in quantum computing. For three-qubit systems, there are three types of pure states: separable or completely separable, biseparable and genuine entangled states [8–13]. A state is separable if the three-qubit state vector is the tensor product of three one-qubit state vectors. If a three-qubit state can be represented as the tensor product of the state vector of two entangled qubits and the state vector of the third qubit, then we have a biseparable state. States that are neither separable nor biseparable are called genuine entangled. For a three-qubit system, there are two nonequivalent classes of truly entangled states, the so-called Greenberger–Horn–Zeilinger states (GHZ-states) and Werner states (W-tates). For three-qubit mixed states, there are also separable, biseparable, or genuine entangled states. Genuine entangled GHZ-type states can be used for deterministic teleportation or dense coding, and W-type qubit states are used in quantum information processing. In recent years, entangled states of the GHZ- and W-type have been obtained experimentally for three-qubit systems of various physical natures: superconducting Josephson rings, ions in magnetic traps, quantum dots, etc. (see links in [1–17]). Electromagnetic fields of cavities are usually used to generate, monitor and control entangled states of qubits. Accordingly, to theoretically describe the dynamics of entanglement of qubits interacting with the electromagnetic fields of cavities, the Tavis–Cummings model and its generalizations are used (see links in [18]). Recently, much attention has been paid to studying the dynamics of qubits interacting with the thermal fields of cavities. This is due to the fact that thermal photons are always present in cavities, which have finite temperatures. Depending on the physical nature of the qubits, the temperatures of the cavities vary from nano- and millikelvins to room temperatures. Interaction with the thermal fields of cavities leads to Rabi oscillations of qubit entanglement parameters, which degrades the quality of quantum information stored in the qubit subsystem and leads to the appearance of additional errors. This means that it is required to study in detail the influence of the thermal noise of the cavity on the dynamics of entanglement of qubits prepared initially in entangled states. As a result, a large number of works have recently been devoted to studying the dynamics of qubit entanglement induced by thermal noise of the cavity (see references in [7,19]). Another problem that arises when qubits interact with the thermal fields of cavities — is the appearance of the effect of sudden death of entanglement, i.e., the disappearance of qubit entanglement at times significantly shorter than the decoherence time. This effect was theoretically predicted by Yu and Eberly [20] while studying the unitary dynamics of two qubits in a cavities. Later, this effect was observed experimentally [21]. For a three-qubit system, the opportunity of instantaneous death of entanglement of three qubits interacting with the thermal field of a common cavity was predicted in the work [22].

In the works [23,24] the three-qubit Tavis–Cummings model was studied, in which two qubits are locked in a cavity and interact with the general field of this cavity, and the third qubit is in a free state. As a quantitative measure of entanglement, the authors chose the negativity of all potential pairs of qubits in a three-qubit system. Meanwhile, in the work [23] the authors limited themselves to considering the dynamics of the system in the case of initial biseparable states of qubits and the vacuum state of the field, and in the work [24] — initial genuine entangled W-type states and Fock field states. It is of interest to generalize the results of the works [23,24] to the case when the qubits in the cavity interact with the general thermal field.

In this work, we considered a system of three identical qubits, two of which are locked in a microwave cavity and interact with the thermal field mode, and the third qubit can move freely outside the cavity. Truly entangled W-type states were chosen as the initial states of the qubit subsystem. We found an exact solution to the evolution equation for the system under consideration, and based on the exact solution, we calculated the negativity of pairs of qubits. The difference in the behavior of entanglement parameters for different types of W-states is also shown.

1. Model and its exact solution

Let us review a system consisting of three identical qubits A, B and C. Two qubits B and C interact resonantly with the quantized electromagnetic field of the cavity. The A qubit can move freely outside the cavity. The Hamiltonian of the interaction of such a system in the dipole approximation and the rotating wave approximation can be written in the form

$$\hat{H}_I = \sum_{i=2}^{3} \hbar\gamma \left(\hat{\sigma}_i^+ \hat{a} + \hat{\sigma}_i^- \hat{a}^+\right), \tag{1}$$

where $\hat{\sigma}_i^+ = |+\rangle_i\langle-|$ and $\hat{\sigma}_i^- = |-\rangle_i\langle+|$ — are raising and lowering operators in the i qubit, $\hat{a}$ and $\hat{a}^+$ are operators of the destruction and creation of photons in the cavity mode, γ is a constant that characterizes the qubit-photon interaction. We assume that all A, B and C qubits have the same energy gaps and the interaction constants of the B and C qubits with the cavity are the same.

We will assume that at the initial moment of time the qubits are in one of the following genuine entangled W-type states:

$$|\Psi(0)\rangle_{ABC} = \cos\theta|+,+,-\rangle + \sin\theta\sin\varphi|+,-,+\rangle + \sin\theta\cos\varphi|-,+,+\rangle, \tag{2}$$

$$|\Psi(0)\rangle_{ABC} = \cos\theta|-,-,+\rangle + \sin\theta\sin\varphi|-,+,-\rangle + \sin\theta\cos\varphi|+,-,-\rangle, \tag{3}$$

where the θ and φ determine the initial degree of entanglement of qubits. As the initial state of the cavity field, we choose a single-mode thermal field, the density matrix of which is expressed by the formula

$$\rho_F(0) = \sum_n p_n|n\rangle\langle n|. \tag{4}$$

Here the statistical weight of p_n is as follows:

$$p_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}},$$

where $\bar{n}$ — the average number of thermal photons, which is determined by the standard Bose–Einstein formula:

$$\bar{n} = \left(\exp[\hbar\omega/k_{\mathrm{B}}T] - 1\right)^{-1}.$$

Here k_{B} — Boltzmann constant, T — cavity temperature.

Let us first find the time evolution of the system under consideration for the Fock initial states of the field n ($n = 0, 1, 2, \ldots$), and then generalize the resulting solution to the case of a thermal field. We have solved the nonstationary Schrodinger equation for initial states (2)–(3):

$$i\hbar\frac{\partial|\Psi_n(t)\rangle}{\partial t} = \hat{H}_I|\Psi_n(t)\rangle, \tag{5}$$

where $|\Psi_n(t)\rangle$ — wave function describing the state of the system, which includes qubits and the cavity field mode, at an arbitrary moment in time t. In this case, the solution to equation (5) should be sought separately for different numbers of photons in the mode n.

Let us write down the explicit form of the wave function for the initial state of qubits (2) and the number of photons in the mode $n = 0$:

$$\begin{aligned}|\Psi_{n=0}(t)\rangle &= X_1(t)|+,+,-,0\rangle + X_2(t)|+,-,+,0\rangle \\ &+ X_3(t)|+,-,-,1\rangle + Z_1(t)|-,+,+,0\rangle \\ &+ Z_2(t)|-,+,-,1\rangle + Z_3(t)|-,-,+,1\rangle \\ &+ Z_4(t)|-,-,-,2\rangle.\end{aligned} \tag{6}$$

In the case of the initial number of photons in the mode $n \geq 1$:

$$\begin{aligned}|\Psi_{n\geq1}(n,t)\rangle &= B_1(n,t)|+,+,+,n-1\rangle \\ &+ B_2(n,t)|+,+,-,n\rangle + B_3(n,t)|+,-,+,n\rangle \\ &+ B_4(n,t)|+,-,-,n+1\rangle + G_1(n,t)|-,+,+,n\rangle \\ &+ G_2(n,t)|-,+,-,n+1\rangle + G_3(n,t)|-,-,+,n+1\rangle \\ &+ G_4(n,t)|-,-,-,n+2\rangle.\end{aligned} \tag{7}$$

Substituting the wave functions (6)–(7) and the interaction Hamiltonian (1) into the nonstationary Schrodinger equation (5), we obtain the following systems of differential equations:

$$\begin{cases} i\dot{X}_1(t) = \gamma X_3(t), \\ i\dot{X}_2(t) = \gamma X_3(t), \\ i\dot{X}_3(t) = \gamma\left(X_1(t) + X_2(t)\right), \end{cases}$$

$$\begin{cases} i\dot{B}_1(n,t) = \gamma\sqrt{n}\left(B_3(n,t) + B_2(n,t)\right), \\ i\dot{B}_2(n,t) = \gamma\left(\sqrt{n+1}\,B_4(n,t) + \sqrt{n}B_1(n,t)\right), \\ i\dot{B}_3(n,t) = \gamma\left(\sqrt{n+1}\,B_4(n,t) + \sqrt{n}B_1(n,t)\right), \\ i\dot{B}_4(n,t) = \gamma\sqrt{n+1}\,\left(B_3(n,t) + B_2(n,t)\right), \end{cases} \quad (8)$$

$$\begin{cases} i\dot{G}_1(n,t) = \gamma\sqrt{n+1}\,\left(G_3(n,t) + G_2(n,t)\right), \\ i\dot{G}_2(n,t) = \gamma\left(\sqrt{n+2}\,G_4(n,t) + \sqrt{n+1}\,G_1(n,t)\right), \\ i\dot{G}_3(n,t) = \gamma\left(\sqrt{n+2}\,G_4(n,t) + \sqrt{n+1}\,G_1(n,t)\right), \\ i\dot{G}_4(n,t) = \gamma\sqrt{n+2}\,\left(G_3(n,t) + G_2(n,t)\right). \end{cases} \quad (9)$$

Solving systems of differential equations (8),(9) taking into account the initial conditions

$$X_1(0) = \cos\theta,\ X_2(0) = \sin\theta\sin\varphi,\ X_3(0) = 0;$$

$$B_2(0) = \cos\theta,\ B_3(0) = \sin\theta\sin\varphi,\ B_1(0) = B_4(0) = 0;$$

$$G_1(0) = \sin\theta\cos\varphi,\ G_2(0) = G_3(0) = G_4(0) = 0$$

and taking into account that $G_i(n,t) \to Z_i(t)$ with the number of photons in the mode $n = 0$, we obtain analytical expressions for all time coefficients:

$$X_1(t) = \cos^2\left(\frac{\gamma t}{\sqrt{2}}\right)\cos\theta - \sin^2\left(\frac{\gamma t}{\sqrt{2}}\right)\sin\theta\sin\varphi,$$

$$X_2(t) = \cos^2\left(\frac{\gamma t}{\sqrt{2}}\right)\sin\theta\sin\varphi - \sin^2\left(\frac{\gamma t}{\sqrt{2}}\right)\cos\theta,$$

$$X_3(t) = -\frac{i\sin(\sqrt{2}\gamma t)(\cos\theta + \sin\theta\sin\varphi)}{\sqrt{2}},$$

$$B_1(n,t) = -\frac{i\sqrt{n}\sin(\gamma t\sqrt{4n+2})(\cos\theta + \sin\theta\sin\varphi)}{\sqrt{4n+2}},$$

$$B_2(n,t) = \frac{1}{2}\Big[\left(1 + \cos(\gamma t\sqrt{4n+2})\right)\cos\theta + (\cos(\gamma t\sqrt{4n+2}) - 1)\sin\theta\sin\varphi\Big],$$

$$B_3(n,t) = \frac{1}{2}\Big[(\cos(\gamma t\sqrt{4n+2}) - 1)\cos\theta + (1 + \cos(\gamma t\sqrt{4n+2}))\sin\theta\sin\varphi\Big], \quad (10)$$

$$B_4(n,t) = -\frac{i\sqrt{n+1}\sin(\gamma t\sqrt{4n+2})(\cos\theta + \sin\theta\sin\varphi)}{\sqrt{4n+2}},$$

$$G_1(n,t) = \frac{\left[2 + n + (n+1)\cos(\gamma t\sqrt{4n+6})\right]\cos\varphi\sin\theta}{2n+3},$$

$$G_2(n,t) = G_3(n,t) = -\frac{i\sqrt{n+1}\cos\varphi\sin\theta\sin(\gamma t\sqrt{4n+6})}{\sqrt{4n+6}},$$

$$G_4(n,t) = -\frac{2\sqrt{n+1}\sqrt{n+2}\cos\varphi\sin\theta\sin^2\left(\gamma t\sqrt{\frac{3}{2}+n}\right)}{2n+3}.$$

Let us carry out similar calculations for another initial entangled state of the W-type (3). Let us write down the explicit form of the wave function for the initial state of qubits (3) at subsequent moments of time and the number of photons in the mode $n = 0$:

$$|\Psi_{n=0}(t)\rangle = \sin\theta\cos\varphi|+,-,-,0\rangle + Y_1(t)|-,+,-,0\rangle + Y_2(t)|-,-,+,0\rangle + Y_3(t)|-,-,-,1\rangle. \quad (11)$$

In the case of the number of photons in the mode $n = 1$:

$$|\Psi_{n=1}(t)\rangle = X_1(t)|+,+,-,0\rangle + X_2(t)|+,-,+,0\rangle + X_3(t)|+,-,-,1\rangle + Z_1(t)|-,+,+,0\rangle + Z_2(t)|-,+,-,1\rangle + Z_3(t)|-,-,+,1\rangle + Z_4(t)|-,-,-,2\rangle. \quad (12)$$

In the case of the number of photons in the mode $n \geq 2$:

$$|\Psi_{n\geq 2}(n,t)\rangle = B_1(n,t)|+,+,+,n-2\rangle + B_2(n,t)|+,+,-,n-1\rangle + B_3(n,t)|+,-,+,n-1\rangle + B_4(n,t)|+,-,-,n\rangle + G_1(n,t)|-,+,+,n-1\rangle + G_2(n,t)|-,+,-,n\rangle + G_3(n,t)|-,-,+,n\rangle + G_4(n,t)|-,-,-,n+1\rangle. \quad (13)$$

Substituting the wave functions (11)–(13) and the interaction Hamiltonian (1) into the nonstationary Schrodinger equation (5), we obtain the following systems of differential equations:

$$\begin{cases} i\dot{Y}_1(t) = \gamma Y_3(t), \\ i\dot{Y}_2(t) = \gamma Y_3(t), \\ i\dot{Y}_3(t) = \gamma\left(Y_1(t) + Y_2(t)\right), \end{cases}$$

$$\begin{cases} i\dot{B}_1(n,t) = \gamma\sqrt{n-1}\,\left(B_3(n,t) + B_2(n,t)\right), \\ i\dot{B}_2(n,t) = \gamma\left(\sqrt{n}\,B_4(n,t) + \sqrt{n-1}\,B_1(n,t)\right), \\ i\dot{B}_3(n,t) = \gamma\left(\sqrt{n}\,B_4(n,t) + \sqrt{n-1}\,B_1(n,t)\right), \\ i\dot{B}_4(n,t) = \gamma\sqrt{n}\,\left(B_3(n,t) + B_2(n,t)\right). \end{cases} \quad (14)$$

$$\begin{cases} i\dot{X}_1(t) = \gamma X_3(t), \\ i\dot{X}_2(t) = \gamma X_3(t), \\ i\dot{X}_3(t) = \gamma\left(X_1(t) + X_2(t)\right), \end{cases}$$

$$\begin{cases} i\dot{G}_1(n,t) = \gamma\sqrt{n}\left(G_3(n,t) + G_2(n,t)\right), \\ i\dot{G}_2(n,t) = \gamma\left(\sqrt{n+1}G_4(n,t) + \sqrt{n}G_1(n,t)\right), \\ i\dot{G}_3(n,t) = \gamma\left(\sqrt{n+1}G_4(n,t) + \sqrt{n}G_1(n,t)\right), \\ i\dot{G}_4(n,t) = \gamma\sqrt{n+1}\left(G_3(n,t) + G_2(n,t)\right). \end{cases} \quad (15)$$

Solving systems of differential equations (14),(15) taking into account the initial conditions

$$Y_1(0) = \sin\theta\sin\varphi,\ Y_2(0) = \cos\theta,\ Y_3(0) = 0;$$

$$X_3(0) = \sin\theta\cos\varphi,\ X_1(0) = X_2(0) = 0;$$

$$B_4(0) = \sin\theta\cos\varphi,\ B_1(0) = B_2(0) = B_3(0) = 0;$$

$$G_2(0) = \sin\theta\sin\varphi,\ G_3(0) = \cos\theta,\ G_1(0) = G_4(0) = 0$$

and taking into account that $G_i(n,t) \to Z_i(t)$ with the number of photons in the mode $n = 1$, we obtain analytical expressions for all time coefficients:

$$Y_1(t) = \cos^2\left(\frac{\gamma t}{\sqrt{2}}\right)\sin\theta\sin\varphi - \sin^2\left(\frac{\gamma t}{\sqrt{2}}\right)\cos\theta,$$

$$Y_2(t) = \cos^2\left(\frac{\gamma t}{\sqrt{2}}\right)\cos\theta - \sin^2\left(\frac{\gamma t}{\sqrt{2}}\right)\sin\theta\sin\varphi,$$

$$Y_3(t) = -\frac{i\sin(\sqrt{2}\gamma t)\,(\cos\theta + \sin\theta\sin\varphi)}{\sqrt{2}},$$

$$X_1(t) = X_2(t) = -\frac{i\cos\varphi\sin(\sqrt{2}\gamma t)\sin\theta}{\sqrt{2}},$$

$$X_3(t) = \cos(\sqrt{2}\gamma t)\cos\varphi\sin\theta,$$

$$B_1(n,t) = -\frac{2\sqrt{n-1}\sqrt{n}\cos\varphi\sin\theta\sin^2\left(\gamma t\sqrt{n-\frac{1}{2}}\right)}{2n-1},$$

$$B_2(n,t) = B_3(n,t) = -\frac{i\sqrt{n}\cos\varphi\sin\theta\sin(\gamma t\sqrt{4n-2})}{\sqrt{4n-2}},$$

$$B_4(n,t) = \frac{(n-1+n\cos(\gamma t\sqrt{4n-2}))\cos\varphi\sin\theta}{2n-1},$$

$$G_1(n,t) = -\frac{i\sqrt{n}\sin(\gamma t\sqrt{4n+2})(\cos\theta + \sin\theta\sin\varphi)}{\sqrt{4n+2}},$$

$$G_2(n,t) = \frac{1}{2}\Big[(\cos(\gamma t\sqrt{4n+2}) - 1)\cos\theta + (1 + \cos(\gamma t\sqrt{4n+2}))\sin\theta\sin\varphi\Big],$$

$$G_3(n,t) = \frac{1}{2}\Big[(1 + \cos(\gamma t\sqrt{4n+2}))\cos\theta + (\cos(\gamma t\sqrt{4n+2}) - 1)\sin\theta\sin\varphi\Big],$$

$$G_4(n,t) = -\frac{i\sqrt{n+1}\sin(\gamma t\sqrt{4n+2})(\cos\theta + \sin\theta\sin\varphi)}{\sqrt{4n+2}}. \quad (16)$$

To calculate any known entanglement criteria for three-qubit systems, we need to calculate reduced two- and three-qubit density matrices. As a first step to realize this goal, it is required to calculate the density matrix of the complete system „three qubits + field mode". Knowing the explicit form of the time wave functions, we can construct the density matrix of the complete system as:

$$\rho_{ABCF}(t) = \sum_{n=0}^{\infty} p_n|\Psi_n(t)\rangle\langle\Psi_n(t)|, \quad (17)$$

which for state (2) has the form

$$\rho_{ABCF}(t) = \sum_{n=1}^{\infty} p_n|\Psi_{n\geq 1}(n,t)\rangle\langle\Psi_{n\geq 1}(n,t)| + p_0|\Psi_{n=0}(t)\rangle\langle\Psi_{n=0}(t)|, \quad (18)$$

and for the (3)

$$\rho_{ABCF}(t) = \sum_{n=2}^{\infty} p_n|\Psi_{n\geq 2}(n,t)\rangle\langle\Psi_{n\geq 2}(n,t)| + p_1|\Psi_{n=1}(t)\rangle\langle\Psi_{n=1}(t)| + p_0|\Psi_{n=0}(t)\rangle\langle\Psi_{n=0}(t)|. \quad (19)$$

Currently, strict quantitative entanglement criteria are established only for two-qubit systems. These include consistency [25] and negativity [26,27]. In this work, to assess the degree of entanglement, we will calculate the negativity of pairs of qubits included in a three-qubit system.

To calculate the criterion for the negativity of the pair of qubits i and j, we will need a two-qubit density matrix $\rho_{ij}(t)$, which is defined as follows:

$$\rho_{ij}(t) = Tr_k Tr_F \rho_{ABCF} \quad (i, j, k = A, B, C;\ i \neq j \neq k). \quad (20)$$

As is known, negativity is given by the following formula:

$$\varepsilon_{ij} = -2\sum_l (\mu_{ij})_l^-, \quad (21)$$

where μ_{ij} — negative eigenvalues of the reduced two-qubit density matrix $\rho_{ij}^T(t)$ partially transposed over the variables of one qubit, which has the following form for states (2),(3):

$$\rho_{ij}^T(t) = \begin{pmatrix} \rho_{11}^{ij} & 0 & 0 & \left(\rho_{23}^{ij}\right)^* \\ 0 & \rho_{22}^{ij} & 0 & 0 \\ 0 & 0 & \rho_{33}^{ij} & 0 \\ \rho_{23}^{ij} & 0 & 0 & \rho_{44}^{ij} \end{pmatrix}. \quad (22)$$

Then the formula for negativity (21) is transformed into the following expression:

$$\varepsilon_{ij} = \sqrt{(\rho_{44}^{ij} - \rho_{11}^{ij})^2 + 4\rho_{23}^{ij}} - \rho_{11}^{ij} - \rho_{44}^{ij}. \quad (23)$$

For the initial state (2) and qubits A and B, the elements of the density matrix are expressed as follows:

$$\rho_{11}^{AB}(t) = \sum_{n=1}^{\infty} p_n\left[|B_1(n,t)|^2 + |B_2(n,t)|^2\right] + p_0|X_1(t)|^2,$$

$$\rho_{22}^{AB}(t) = \sum_{n=1}^{\infty} p_n \left[|B_3(n,t)|^2 + |B_4(n,t)|^2\right] + p_0[|X_2(t)|^2 + |X_3(t)|^2],$$

$$\rho_{33}^{AB}(t) = \sum_{n=1}^{\infty} p_n \left[|G_1(n,t)|^2 + |G_2(n,t)|^2\right] + p_0 \left[|Z_1(t)|^2 + |Z_2(t)|^2\right],$$

$$\rho_{44}^{AB}(t) = \sum_{n=1}^{\infty} p_n \left[|G_3(n,t)|^2 + |G_4(n,t)|^2\right] + p_0 \left[|Z_3(t)|^2 + |Z_4(t)|^2\right],$$

$$\rho_{23}^{AB}(t) = \sum_{n=1}^{\infty} p_n \left[B_3(n,t)G_1^*(n,t) + B_4(n,t)G_2^*(n,t)\right] + p_0 \left[X_3(t)Z_2^*(t) + X_2(t)Z_1^*(t)\right].$$

For the initial state (2) and qubits B and C, the elements of the density matrix are expressed as follows:

$$\rho_{11}^{BC}(t) = \sum_{n=1}^{\infty} p_n \left[|B_1(n,t)|^2 + |G_1(n,t)|^2\right] + p_0|Z_1(t)|^2,$$

$$\rho_{22}^{BC}(t) = \sum_{n=1}^{\infty} p_n \left[|B_2(n,t)|^2 + |G_2(n,t)|^2\right] + p_0[|X_1(t)|^2 + |Z_2(t)|^2],$$

$$\rho_{33}^{BC}(t) = \sum_{n=1}^{\infty} p_n \left[|B_3(n,t)|^2 + |G_3(n,t)|^2\right] + p_0 \left[|X_2(t)|^2 + |Z_3(t)|^2\right],$$

$$\rho_{44}^{BC}(t) = \sum_{n=1}^{\infty} p_n \left[|B_4(n,t)|^2 + |G_4(n,t)|^2\right] + p_0 \left[|X_3(t)|^2 + |Z_4(t)|^2\right],$$

$$\rho_{23}^{BC}(t) = \sum_{n=1}^{\infty} p_n \left[B_2(n,t)B_3^*(n,t) + G_2(n,t)G_2^*(n,t)\right] + p_0 \left[X_1(t)X_2^*(t) + Z_2(t)Z_3^*(t)\right].$$

For the initial state (3) and qubits A and B, the elements of the density matrix are as follows:

$$\rho_{11}^{AB}(t) = \sum_{n=2}^{\infty} p_n \left[|B_1(n,t)|^2 + |B_2(n,t)|^2\right] + p_1|X_1(t)|^2,$$

$$\rho_{22}^{AB}(t) = \sum_{n=2}^{\infty} p_n \left[|B_3(n,t)|^2 + |B_4(n,t)|^2\right] + p_1 \left[|X_2(t)|^2 + |X_3(t)|^2\right] + p_0(\sin\theta\cos\varphi)^2,$$

$$\rho_{33}^{AB}(t) = \sum_{n=2}^{\infty} p_n \left[|G1(n,t)|^2 + |G_2(n,t)|^2\right] + p_1[|Z_1(t)|^2 + |Z_2(t)|^2] + p_0|Y_1(t)|^2,$$

$$\rho_{44}^{AB}(t) = \sum_{n=2}^{\infty} p_n \left[|G_3(n,t)|^2 + |G_4(n,t)|^2\right] + p_1 \left[|Z_3(t)|^2 + |Z_4(t)|^2\right] + p_0 \left[|Y_2(t)|^2 + |Y_3(t)|^2\right],$$

$$\rho_{23}^{AB}(t) = \sum_{n=2}^{\infty} p_n \left[B_4(n,t)G_2^*(n,t) + B_3(n,t)G_1^*(n,t)\right] + p_1 \left[X_2(t)Z_1^*(t) + X_3(t)Z_2^*(t)\right] + p_0 \sin\theta\cos\varphi Y_1^*(t).$$

For the initial state (3) and qubits B and C, the elements of the density matrix are as follows:

$$\rho_{11}^{BC}(t) = \sum_{n=2}^{\infty} p_n \left[|B_1(n,t)|^2 + |G_1(n,t)|^2\right] + p_1|Z_1(t)|^2,$$

$$\rho_{22}^{BC}(t) = \sum_{n=2}^{\infty} p_n \left[|B_2(n,t)|^2 + |G_2(n,t)|^2\right] + p_1 \left[|X_1(t)|^2 + |Z_2(t)|^2\right] + p_0|Y_1(t)|^2,$$

$$\rho_{33}^{BC}(t) = \sum_{n=2}^{\infty} p_n \left[|B_3(n,t)|^2 + |G_3(n,t)|^2\right] + p_1[|X_2(t)|^2 + |Z_3(t)|^2] + p_0|Y_2(t)|^2,$$

$$\rho_{44}^{BC}(t) = \sum_{n=2}^{\infty} p_n \left[|B_4(n,t)|^2 + |G_4(n,t)|^2\right] + p_1 \left[|X_3(t)|^2 + |Z_4(t)|^2\right] + p_0 \left[(\sin\theta\cos\varphi)^2 + |Y_3(t)|^2\right],$$

$$\rho_{23}^{BC}(t) = \sum_{n=2}^{\infty} p_n \left[B_2(n,t)B_3^*(n,t) + G_2(n,t)G_3^*(n,t)\right] + p_1 \left[X_1(t)X_2^*(t) + Z_2(t)Z_3^*(t)\right] + p_0 Y_1(t)Y_2^*(t).$$

2. Results and discussion

The results of computer simulation of the negativity of qubit pairs for the initial states of qubits (2) and (3) and various model parameters are presented in Fig. 1–10.

Figure 1 shows the dependence of the entanglement parameter ε_{AB} of the qubits A and B on the dimensionless time γt for the genuine entangled initial state of qubits (2) with, $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$ and various values of the average number of photons in the cavity mode. The figure clearly shows that at some times the entanglement abruptly disappears and remains zero for a finite time before being reborn. This means there is an sudden death effect of the entanglement in the system. The figure also shows that as the average number of thermal photons increases, the maximum degree of qubit entanglement decreases rapidly.

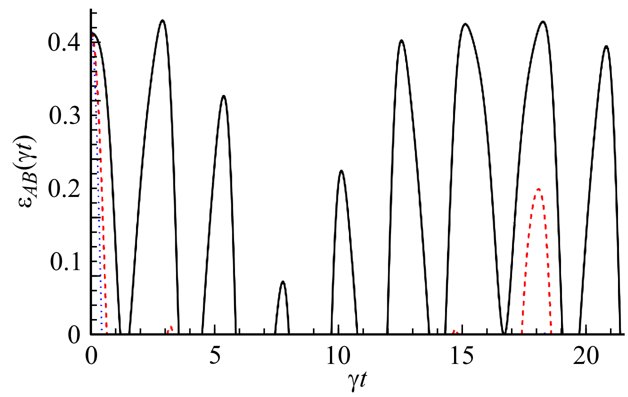

Figure 1. Graph of the dependence of the negativity criterion $\varepsilon_{AB}(\gamma t)$ on the reduced time γt for qubits A and B and the initial genuine entangled state (2). Parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$. Average number of thermal photons in the mode: $\bar{n} = 0.001$ (solid line), $\bar{n} = 1$ (dashed line), $\bar{n} = 2.5$ (dotted line).

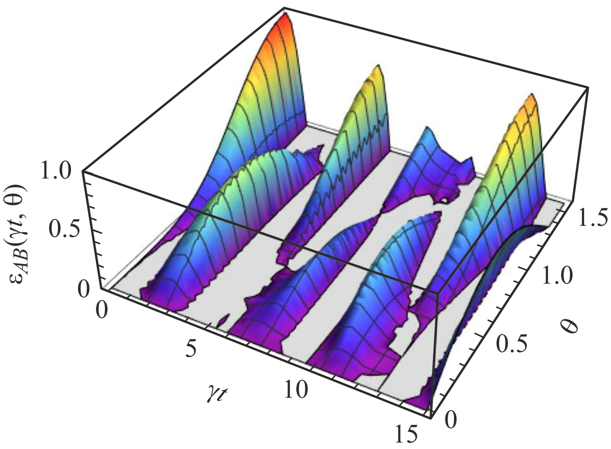

Figure 2. 3D-graph of the dependence of the negativity $\varepsilon_{AB}(\gamma t, \theta)$ on the reduced time γt and the parameter θ for qubits A and B in the case of an initial genuine entangled state (2) and fixed values of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\varphi = \pi/4$.

Let us note that the disappearance of entanglement of a free qubit and a qubit in the cavity with increasing intensity of the thermal field occurs in the three-qubit model under consideration much faster than in the two-qubit model, in which one of the qubits is free and the second is locked in the thermal cavity [28] .

Figure 2 shows 3D- the dependence of the negativity $\varepsilon_{AB}(\gamma t, \theta)$ for the same qubits on the reduced time γt and the parameter θ for the same initial state (2) and fixed values of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\varphi = \pi/4$. Figure 3 shows 3D- the dependence of the negativity $\varepsilon_{AB}(\gamma t, \varphi)$ for the same qubits on the reduced time γt and the parameter φ for the same initial state (2) and fixed values of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\theta = \arccos[1/\sqrt{3}]$. It is clearly seen in Figs. 2 and 3 that for the selected initial state of qubits, sydden death and birth of entanglement take place for any values of the parameters θ and φ. It is also worth noting that the duration of the time intervals between the sudden death and sudden birth of the entanglement strongly depends on the initial parameters θ and φ. The graphs also show that the smaller the φ (Fig. 3) and the larger the θ (Fig. 2), the greater the maximum degree of entanglement. Thus, the length of time without entanglement and the maximum degree of entanglement depend significantly on the initial state of the qubits. Figure 4 shows the dependence of the entanglement parameter of the qubits B and C on the dimensionless time γt for the genuine entangled initial state of the qubits (2), the parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$ and various values of the average number of photons in the cavity mode. It is clearly seen that for the parameter of entanglement of

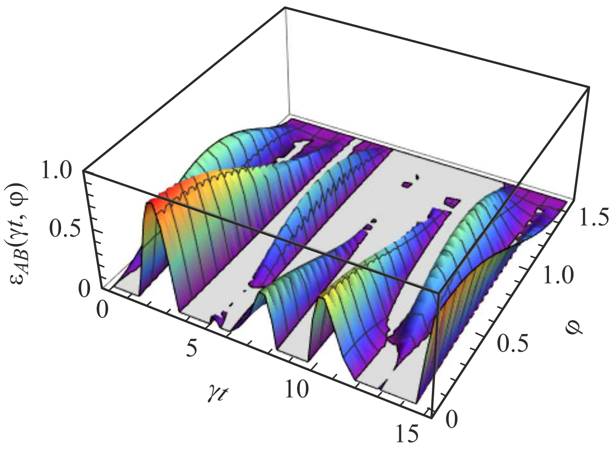

Figure 3. 3D-graph of the dependence of the negativity $\varepsilon_{AB}(\gamma t, \varphi)$ on the reduced time γt and the parameter φ for qubits A and B in the case of an initial genuine entangled state (2) and fixed values of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\theta = \arccos[1/\sqrt{3}]$.

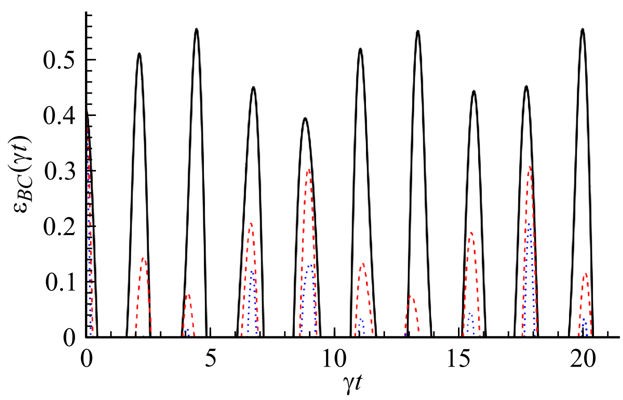

Figure 4. Graph of the dependence of the negativity $\varepsilon_{BC}(\gamma t)$ on the reduced time γt for qubits B and C and the initial genuine entangled state (2). Parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$. Average number of thermal photons: $\bar{n} = 0.001$ (solid line), $\bar{n} = 1$ (dashed line), $\bar{n} = 2.5$ (dashed line).

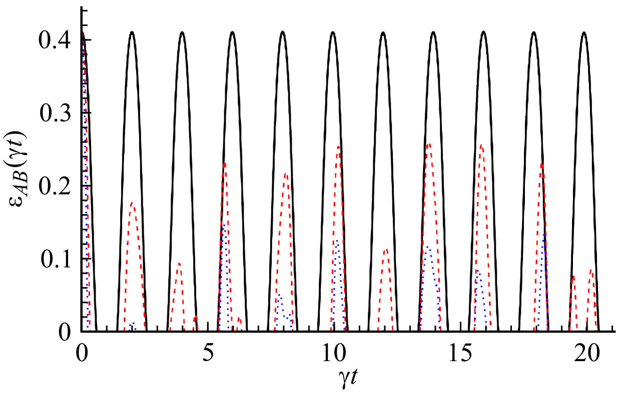

Figure 5. Graph of the dependence of the negativity $\varepsilon_{ij}(\gamma t)$ on the reduced time γt in the model „three qubits in a common cavity" for any pair of qubits and the initial genuine entangled state (2). Parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$. Average number of thermal photons: $\bar{n} = 0.001$ (solid line), $\bar{n} = 1$ (dashed line), $\bar{n} = 2.5$ (dashed line).

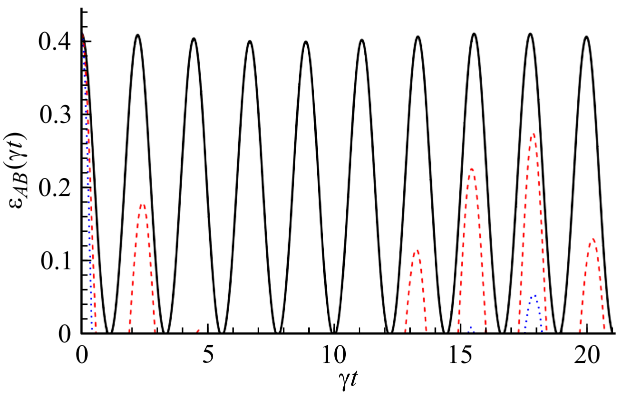

Figure 6. Graph of the dependence of the negativity $\varepsilon_{AB}(\gamma t)$ on the reduced time γt for qubits A and B and the initial genuine entangled state (3). Parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$. The average number of photons: $\bar{n} = 0.001$ (solid line), $\bar{n} = 1$ (dashed line), $\bar{n} = 2.5$ (dashed line).

qubits located inside the cavity, the effect of sudden death of entanglement also occurs. However, unlike the A and B qubits, the B and C qubits turn out to be entangled even in the case of intense thermal fields of the cavity. In this case, the duration of time intervals during which there is no entanglement, decreases. For comparison, Fig. 5 shows the time dependence of the negativity of the qubits A and B (or A and C) for three identical qubits locked in an ideal cavity and resonantly interacting with the general thermal field, for an initial state of the form (2) (the corresponding formulas for the negativity of pairs of qubits are given in our work [7]). A comparison of the figures shows that for the model under consideration, the behavior of the negativity of a pair of qubits locked in a cavity is similar to the behavior of any pair of qubits in a system of three qubits located in a common cavity.

Figure 6 shows the dependence of the entanglement parameter of the qubits A and B on the dimensionless time γt for the genuine entangled initial state of the qubits (3), $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$ and various values of the average number of photons in the cavity mode. The figure shows that for qubits A and B and the initial state of qubits (3), in contrast to the initial state (2), in the case of low intensities of the thermal field, the effect of sudden death of entanglement is absent. This result is consistent with the results of the work [24], in which the dynamics of qubit entanglement was studied as part of the model under consideration for the initial state of qubits of the form (3) and the vacuum state of the cavity field ($\bar{n} \to 0$). Meanwhile, the effect of sudden death of entanglement occurs with an increase in the intensity of the thermal noise of the cavity. It can be seen that the time intervals during which there is no entanglement for different intensities are significantly less than in the case of the initial state of qubits (2). Figure 7 shows 3*D*- the dependence of the negativity $\varepsilon_{AB}(\gamma t, \theta)$ for the same qubits on the reduced time γt and the parameter θ for the same initial state (3) and fixed values of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\varphi = \pi/4$. Figure 8 shows 3*D*- the dependence of the negativity $\varepsilon_{AB}(\gamma t, \varphi)$ for the same qubits on the reduced time γt and the parameter φ for the same initial state (3) and fixed values of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\theta = \arccos[1/\sqrt{3}]$. It is clearly seen in Figs. 7 and 8 that for the selected initial state of the qubits in the case of low intensities of the thermal field of the cavity, the effect of sudden death of entanglement is absent for any values of the parameters θ and φ, which fundamentally distinguishes the behavior of the entanglement parameter of the selected qubits in the situation under consideration from the case when the qubits are initially prepared in state (2).

Figure 9 shows the dependence of the entanglement parameter of the qubits B and C on the dimensionless time γt for the genuine entangled initial state of the

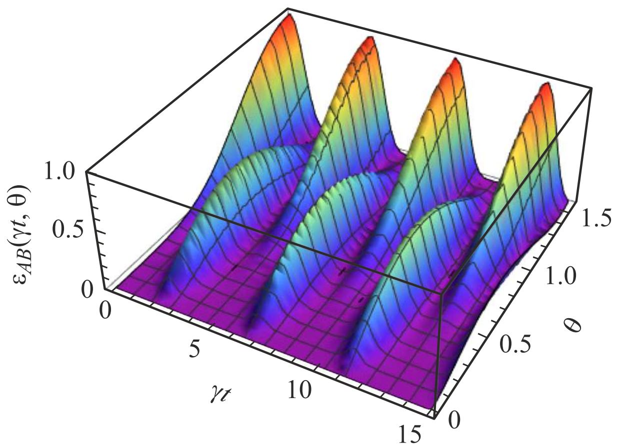

Figure 7. 3*D*-graph of the dependence of the negativity $\varepsilon_{AB}(\gamma t, \theta)$ on the reduced time γt and the parameter θ for qubits *A*and *B* in the case of an initial genuine entangled state (3) and fixed value of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\varphi = \pi/4$.

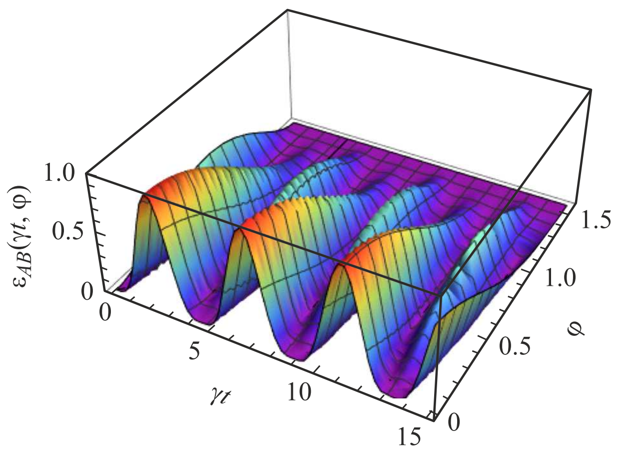

Figure 8. 3D-graph of the dependence of the negativity $\varepsilon_{AB}(\gamma t, \varphi)$ on the reduced time γt and the parameter φ for qubits A and B in the case of an initial genuine entangled state (3) and fixed values of the average number of thermal photons $\bar{n} = 0.001$ and the parameter $\theta = \arccos[1/\sqrt{3}]$.

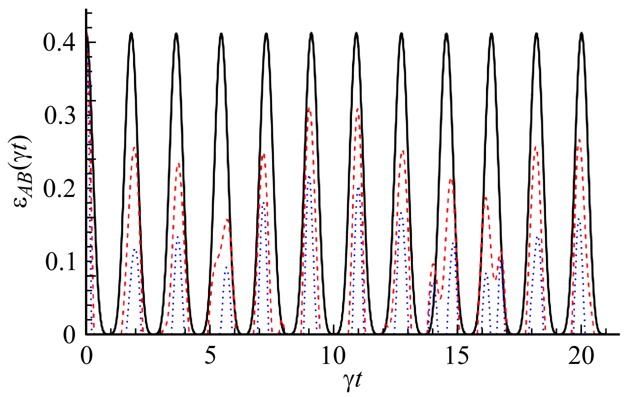

Figure 10. Graph of the dependence of the negativity $\varepsilon_{ij}(\gamma t)$ on the reduced time γt in the model „three qubits in a common cavity“ for any pair of qubits and the initial genuine entangled state (3). Parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$. Average number of thermal photons: $\bar{n} = 0.001$ (solid line), $\bar{n} = 1$ (dashed line), $\bar{n} = 2.5$ (dotted line).

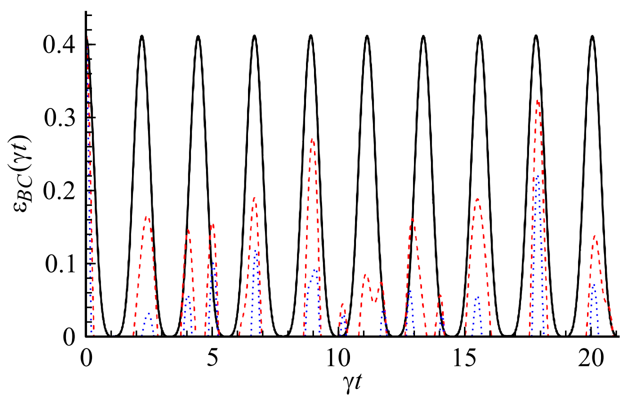

Figure 9. Graph of the dependence of the negativity criterion $\varepsilon_{BC}(\gamma t)$ on the reduced time γt in the model (1) for the second B and third C qubits of the initial genuine entangled state (3) for the initial parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$ with a change in the average number of thermal photons: $\bar{n} = 0.001$ (solid line), $\bar{n} = 1$ (dashed line), $\bar{n} = 2.5$ (dotted line).

qubits (3), the fixed values of the parameters $\varphi = \pi/4$, $\theta = \arccos[1/\sqrt{3}]$ and various values of the average number of photons in the cavity mode. For the selected initial state of the qubits in the case of low thermal field intensities for the B and C qubits, the effect of sudden entanglement death is also absent for any values of the φ, θ parameters, as well as for the A and B qubits. The effect of sudden death of entanglement in the case of the B and C qubits also occurs only with an increase in the intensity of the thermal noise of the cavity. Meanwhile, as in the case of the A and B qubits, the time intervals during which there is no entanglement are significantly shorter for various field intensities than in the case of the initial state of the qubits (2).

Finally, for comparison, Fig. 10 shows the time dependence of the negativity of the qubits A and B (or A and C) for three identical qubits locked in an ideal cavity and ly interacting with the general thermal field, for an initial state of the form (3). Comparison of Figs. 10 and 9 shows that there is a similarity in the behavior of the negativity of a pair of qubits locked in a cavity and any of the pairs of qubits in a three-qubit system in a common cavity. The difference lies in the increase for the last of the considered cases in the maximum negativity values in the peak region and the decrease in the time during which there is no entanglement in the case of intense thermal fields of the cavity. Comparing the behavior of negativity for two initial states of W-type qubits, one can notice that for the model under consideration, the initial genuine entangled state (3) is much more stable with respect to the destructive action of the thermal field for all pairs of qubits than the initial state (2), as is the case in the model with three locked qubits.

Conclusion

In this work, we studied the dynamics of a system of three identical qubits, one of which is in a free state, and the other two are locked in an ideal cavity and interact resonantly with the electromagnetic field mode of this cavity. We obtained an exact solution of the quantum Liouville equation of the considered model for the initial genuine entangled states of W-type qubits and the thermal field of the cavity. Based on the exact solution, analytical expressions are found for the negativity of pairs of qubits: a free qubit-locked qubit and two locked qubits. Calculations are carried out for two genuine entangled normalized W-states of the form (2) and (3), which can be converted from one to the other by means of local operations and classical coupling (LOCC) transformation, and thermal states of the electromagnetic field of the cavity for different average numbers of photons. It is shown that the thermal field of the cavity does not completely destroy the initial entanglement of qubits even for relatively high intensities of the thermal noise of the cavity. It has also been established that for low intensities of the thermal field of the cavity, the effect of sudden death of entanglement occurs only in the case when the qubits are initially prepared in state (2). There is no effect for state (3) under such conditions. As the intensity of the thermal field of the cavity increases, sudden death of qubit entanglement occurs for both the initial state (2) and state (3). Calculations also showed that the duration of the time intervals between sudden death and the resumption of qubit entanglement significantly depends on both the choice of the type of W-state and the degree of initial entanglement of qubits, i.e., on the choice of parameters θ and φ. It has been shown that the initial state of qubits (3) is more resistant to the effects of thermal noise than the initial state (2). This is also true for the model with three locked qubits in a cavity.

For a two-qubit system, in which one of the qubits is free and the second is locked in a thermal cavity, we have previously shown that taking into account the dipole-dipole interaction of qubits, detuning, Kerr nonlinearity and a number of other mechanisms makes it possible to exclude the effect of sudden death of qubit entanglement, which arises due to interaction with the thermal field of the cavity [19,28]. Studying the influence of these mechanisms on the effect of sudden death of entanglement for the model reviewed in this work will be the subject of our subsequent studies.

Conflict of interest

The authors declare that they have no conflict of interest.

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