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Beta poloidal in the „canonical“ Grad–Shafranov equation

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The paper considers the relation between constraint parameter β_p and parameter β which is used in the Grad–Shafranov equation written in the „canonical“ form. It is shown that, when $0 \leqslant \beta_p \leqslant 3.4$, it is possible to use β_p instead of β. When β_p is higher, this dependence gets violated. This effect was assumed to be associated with the fact that, when β_p values are high, in the plasma there appear regions where the plasma poloidal flux, longitudinal current density, and pressure become negative.

Keywords: Grad–Shafanov equation, constraint parameters, plasma pressure, aspect ratio.

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In calculating the tokamak plasma equilibrium, the Grad–Shafanov (GS) equation written in the so-called „canonical“ form is often used [1–5]:

$$\frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = -\mu_0 J_\varphi(\psi, R). \quad (1)$$

Here R is the tokamak major radius, ψ is the poloidal flux function, $\mu_0 = 4\pi \cdot 10^{-7}$ H/m, J_φ is the poloidal flux density defined as

$$J_\varphi = \lambda\left[\frac{R\beta}{R_0}A(\psi, \gamma_1) + R_0\frac{1-\beta}{R}B(\psi, \gamma_2)\right] \quad (2)$$

or

$$J_\varphi = \lambda\left[\frac{R\beta}{R_0} + R_0\frac{1-\beta}{R}\right]\tilde{J}(\psi, \gamma_3). \quad (3)$$

Here parameter λ is associated with normalization of longitudinal plasma current I_p, parameter β is related with parameter β_p that is the ratio of the gas-kinetic plasma pressure to poloidal magnetic field pressure. Papers [2,4–6] state that $\beta \approx \beta_p$, while in [4] it is noticed that the difference between these values does not exceed 3%. $A(\psi, \gamma_1)$, $B(\psi, \gamma_2)$ and $\tilde{J}(\psi, \gamma_3)$ are arbitrary functions. These functions are typically assumed to depend on the normalized quantity $\tilde{\psi} = (\psi - \psi_b)/(\psi_0 - \psi_b)$, where ψ_0 is the function value on the magnetic axis, while ψ_b is that at the plasma column boundary; however, in some cases dependence on $\tilde{\psi} = \psi - \psi_b$ is used [6], parameters γ_1 and γ_2 determine the plasma column internal inductance l_i in (2), and parameter γ_3 determines that in (3).

The GS equation may have many solutions. To select from them the only one that describes the equilibrium of plasma with specific characteristics, it is necessary to introduce several constraint parameters. Most often, those parameters are I_p, β_p and l_i. Sometimes, the value of stability factor in the center or on the boundary of the plasma column is used instead of the value of internal inductance [2].

The constraint parameters are related to parameters used in the GS equation via the following expressions [3] (in Cartesian coordinates):

$$I_p = \int_S J_\varphi(\lambda, \beta, \gamma_i, \tilde{\psi}, x, y)dS, \quad (4)$$

$$\beta_p = \frac{\int_S p(\lambda, \beta, \gamma_i, \tilde{\psi}, x, y)dS}{S\langle B_\theta^2\rangle/2\mu_0} = \frac{0.02}{I_p^2}\int_S p(\lambda, \beta, \gamma_i, \tilde{\psi}, x, y)dS, \quad (5)$$

$$l_i = \frac{2\int_V B_\theta^2 dV}{R_{ax}(\mu_0 I_p)^2}. \quad (6)$$

Angle brackets designate the averaging over the poloidal cross-section area S, $\mu_0 = 4\pi \cdot 10^{-7}$ H/m, V is the volume occupied by plasma, B_θ is the poloidal magnetic field, p is the plasma pressure, R_{ax} is the radius on which the magnetic axis is located. In formula (5), I_p is given in MA, p — in kPa, S — in m^2, magnetic field B_θ — in T, linear dimensions R — in m.

To find solutions to system (4)–(6), it is necessary to solve equation (1) several times with selecting its parameters so that relations (4)–(6) are satisfied.

Thus, if the relationship between β and β_p is known, calculation of the plasma equilibrium is much simpler. Therefore, many researchers make this substitution even if this is rather unreasonable. To our knowledge, almost nobody has systematically studied this relationship up till now.

Fig. 1 presents the results of determining the β dependence on β_p by using the DINA code [7] with different parameters and for different setups with $1 < R < 6$, $0.3 < a < 2$, $0.1 < I_p < 15$, $0.5 < B_\theta < 6$, and extension $1 < K < 2$ with aspect ratio $A \sim 3$. In addition, calculations

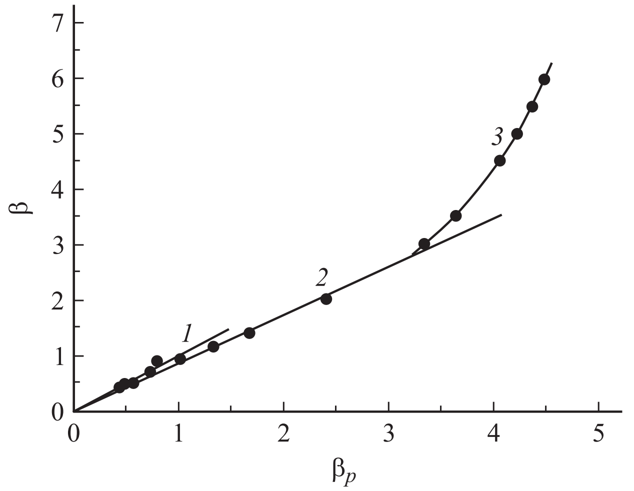

Figure 1. β versus β_p. Relevant comments are given in the text.

obtained for setups with $A \sim 1.5$ and $\beta_p \sim 2$ were used. Calculations were performed for the free-boundary plasma.

Straight line *1* in Fig. 1 is described by expression

$$\beta = 0.99\beta_p \text{ for } 0 \leqslant \beta_p \leqslant 1.5, \tag{7}$$

while straight line *2* is defined as

$$\beta = 0.89\beta_p \text{ for } 0 \leqslant \beta_p \leqslant 3.4. \tag{8}$$

Fig. 1 clearly shows that the linear dependence gets violated as β_p continues increasing (curve *3*).

Notice that accuracy of β_p experimental determination is relatively low. For instance, in [5] it is shown that the β_p measurement error in experiments on setup Doublet III (USA) was 10 to 40%, that on setup KSTAR (Korea) [8] was about 25%.

At present, the main calculations are being performed for plasma whose β_p does not exceed 2–3. At the ITER facility, it is planned to work with plasma with $\beta_p \approx 0.7$ [9]; at DEMO, this parameter will be $\beta_p \approx 1$ [10].

The results of this study show that in these cases it is possible to use in the GS equation β_p instead of β.

Let us consider what occurs at $\beta_p > A$. When β_p reaches the value of aspect ratio A, on the weak field side there gets formed a region where the longitudinal current changes its direction [11]. In this case, the relationship between the poloidal and toroidal β gets violated.

As β_p continues increasing, the plasma boundary is approached by the separatrix that is a surface on which the plasma magnetic flux and pressure are zero [12]; when poloidal β increases further, a region where the pressure becomes formally zero arises in the plasma on the side of weak magnetic field [13]. To illustrate this statement, there were selected such special model parameters at which this effect manifests itself most clearly.

In D–T experiments at TFTR (USA), plasma with $\beta_p \approx 6$ [13] was used.

Fig. 2 illustrates the plasma pressure variation along the equatorial plane. In this figure, variable x is normalized to the plasma minor radius.

Curve *1* was calculated for a tokamak with $A = 1.5$ and $\beta_p = 2$. The figure shows that in this case the plasma pressure is positive in the entire region occupied by plasma. Curve *2* corresponds to $A = 1.5$ and $\beta_p = 6$. Apparently, at such a pressure a region with negative pressure gets formed near the torus internal boundary. Magnetic flux ψ also becomes negative.

The distributions of equi-pressure surfaces are presented in Fig. 3. Fig. 3, *a* corresponds to curve *1* in Fig. 2, and Fig. 3, *b* corresponds to curve *2* in Fig. 2. Apparently, in the case under consideration the negative-pressure region occupies a significant part of the plasma column.

Work [13] shows that, in fact, there is no plasma in the negative-pressure region. Having entered the volume

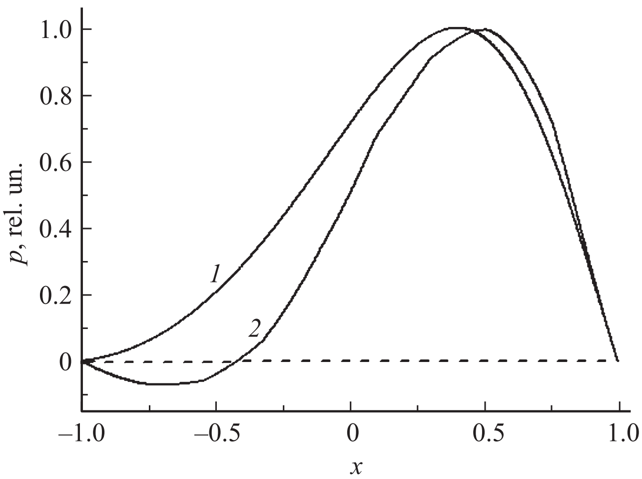

Figure 2. Plasma pressure distributions in the equatorial plane for $\beta_p \cdot \beta_p = 2$ (*1*) and 6(*2*).

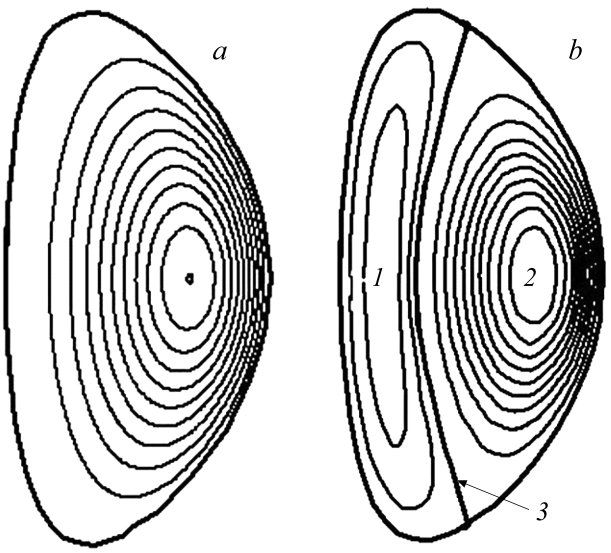

Figure 3. Distribution of equi-pressure surfaces. $\beta_p = 2$ (*a*) and 6 (*b*). *1* — $p < 0$, *2* — $p > 0$, *3* — $p = 0$ (separatrix).

occupied by plasma, the separatrix forms a new plasma boundary, and, hence, the plasma volume decreases.

It is possible that violation of the linear β dependence on β_p is associated with formation in the plasma of regions with negative pressure and negative current density.

The research results allow us to make the following conclusions.

1. 1. If $\beta_p \leqslant 3$, then β_p may be used in calculations instead of β.

2. If $\beta_p > 3$, then this substation is impossible in calculating the plasma equilibrium.

3. If in the plasma there emerge regions where the poloidal magnetic flux, pressure and current density of plasma become negative, then the plasma equilibrium should be calculated only for the region occupied by positive-pressure plasma.

The influence of regions with negative pressure and negative current density on the plasma behavior needs separate consideration.

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Conflict of interests

The authors declare that they have no conflict of interests.

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