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Anisotropy of propagation of spatial surface wave in ferrofluid under the influence of a horizontal magnetic field

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In the linear formulation, the problem of the propagation of a spatial wave on the free surface of a ferrofluid under the influence of a uniform horizontal magnetic field is analytically solved. The resulting formula describes the dependence of the magnetic susceptibility of the ferrofluid on the magnitude of the magnetic field vector. Gravity, magnetic force, and surface tension are taken into account. It is assumed that the wavelength is much smaller than the thickness of the ferrofluid layer. The study investigates how the angle between the wave vector and the magnetic field vector affects both the phase and group velocities of the wave.

Keywords: surface wave, dispersion relation, phase velocity, group velocity, similarity criterion.

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Introduction

Monograph [1] presents the results of multiple studies of spatial linear waves on the free surface of a normal liquid. Regarding electrical conductive fluid paper [2] theoretically shows the presence of anisotropy of surface wave propagation. To study the anisotropy the method of experiment using mercury was suggested.

The anisotropy of surface wave propagation in electrical conductive fluid originates in case when through fluid the applied from power source the horizontal electric current passes having constant density, and homogeneous horizontal magnetic field is applied, and created by them Lorentz force is directed downwards. The anisotropy of wave propagation is provided by perturbation of the Lorentz force caused by change in the shape of the free surface. Study of this phenomenon considering the magnetic field induced by the electric current passing through fluid is given in [3].

Paper [4] plays the main role in development of ferrohydrodynamics. In particular it states the dispersion relation for spatial internal waves on interface of homogeneous ferrofluids with different densities, which fill the region between horizontal plates upon presence of homogeneous horizontal magnetic field. This dispersion relation after simplification was used further to compare with the dispersion relation obtained in present paper. Based on simplified dispersion relation the phase velocity of wave was determined.

To describe phenomena in ferrofluids the equations given in [5,6] are widely used.

In [7] the spatial waves on surface of the incompressible fluid are discussed, it can be magnetized inhomogeneously and isotropically in the external magnetic field. Fluid occupies the bottom half-space. It is assumed that in the studied fluid magnetized by the external field, the magnetic field is induced due to its properties as „liquid magnet“. The magnetic field is induced also in filled with air region above the fluid. We studied the effect of both vertical and horizontal components of the magnetic field on the horizontal flat free surface. It was shown that field tangent to the unperturbed flat surface increases the phase velocity of surface waves, if this field is not perpendicular to the wave front. At that the group velocity also increases. Field with large strength normal to free surface destabilizes the flat free surface. If the wave front is perpendicular to the tangent of flat free surface of magnetic field, then the field does not affect the phase and group velocities of wave. Waves perpendicular to the tangential component of the external field are the most dangerous in view of stability deterioration of the free surface.

Present paper discusses the harmonic wave propagating in horizontal direction [8] in ferrofluid located in applied homogeneous horizontal magnetic field. The hypothesis of „liquid magnet“, like in [5,6], is not used.

The harmonic waves refer to class of linear dispersive waves [9]. We studied wave which length is small as compared to thickness of fluid layer. If there is no wave the ferrofluid is in hydrostatic equilibrium state.

It is assumed that the wave vector $\mathbf{k}$ forms angle θ with vector of magnetic field $\mathbf{H}_0$, occurring at moment of electric magnet switching on. As opposed to [7] the effect of angle θ on phase and group velocities of waves was studied, as well as on equation of free surface in system of coordinates with axis x parallel to vector $\mathbf{H}_0$.

A close (to that considered below) question about the picture of stationary waves caused on the surface of ferrofluid by non-magnetizable obstacle moving in the ferrofluid was studied theoretically and experimentally in [10]. It was found that vertical magnetic field narrows the cone of stationary waves and increases their amplitudes. At that behind the obstacle in area of trace for the linear magnetized ferrofluid the critical value of magnetic field strength, determining occurrence of Rosenzweig instability, decreases. The magnetic field parallel to the obstacle velocity spread the waves cone and decreases their amplitudes until stationary waves suppression. The horizontal field perpendicular to obstacle velocity spreads the waves cone and stabilizes amplitudes.

1. Problem formulation

We study the surface wave in ferrofluid filling the container with non-magnetizable walls. With switched on electrical magnet and wave absence the free surface of the ferrofluid, magnetization vector $\mathbf{M}_0 = \chi(H_0)\mathbf{H}_0$ and vector of magnetic induction $\mathbf{B}_{01} = \mu_0(\mathbf{H}_0 + \mathbf{M}_0) = \mu\mathbf{H}_0$ are horizontal. Here $\mu_0 = 4\pi \cdot 10^{-7}$ H/m — magnetic constant, $\mu(H_0) = \mu_0[1 + \chi(H_0)]$ — magnetic permeability of ferrofluid. There is no magnetic force due to homogeneity of the magnetic field. In atmosphere air above the ferrofluid $\mathbf{B}_{02} = \mu_0\mathbf{H}_0$.

See paper [11], in which experimental study of turbulence was performed, the turbulence is caused by waves on surface of ferrofluid interfaced with air and staying in homogeneous horizontal magnetic field. Aqueous ferrofluid was prepared without use of the stabilizing organic substance as per procedure suggested in paper [12]. The solution is stabilized due to the electric charges of colloidal particles causing the mutual repulsion of particles. The ferrofluid is sensitive to ionic composition of carrying medium. Stabilization is implemented under the condition that counterions in solution are weakly polarizable ions $N(CH_3)_4^+$ and ClO_4^-.

As example let's take from [11] numerical values determining parameters of ferrofluid: coefficient of surface tension $\alpha = 0.059$ N/m, density $\rho = 1324\,\text{kg/m}^3$, initial magnetic susceptibility $\chi_I = 0.69$, saturation magnetization $M_s = 16.9$ kA/m.

Using Langevin function [13] $L(a) = \operatorname{cth} a - 1/a$ of the modified argument $a = \sigma H_0$, where $\sigma = 3\chi_I/M_s$, we approximate according to [14] the experimental magnetization curve of curve

$$M_0(H_0) = M_s L(\sigma H_0). \quad (1)$$

here $\chi_I = \chi(H_0)$ at $H_0 \to 0$, $M_s = M_0(H_0)$ at $H_0 \to \infty$.

At that we have $\chi(H_0) = \frac{M_s}{H_0} L(\sigma H_0)$. Graph of the function $\chi = \chi(H_0)$ is shown in Fig. 1.

Let's introduce Cartesian rectangular coordinate system x, y, z (Fig. 2). Origin will be placed on flat free surface in the absence of wave. Vector $\mathbf{H}_0$ and axis x have same direction, and axis z is directed opposite to vector of gravity acceleration $\mathbf{g}$. Let's designate as $\theta \in [0, \pi/2]$ the angle between vector $\mathbf{H}_0$ and wave vector $\mathbf{k} = (k_x, k_y, 0)$. In the discussed problem $\theta = \text{const}$ is [15] one of similarity criteria.

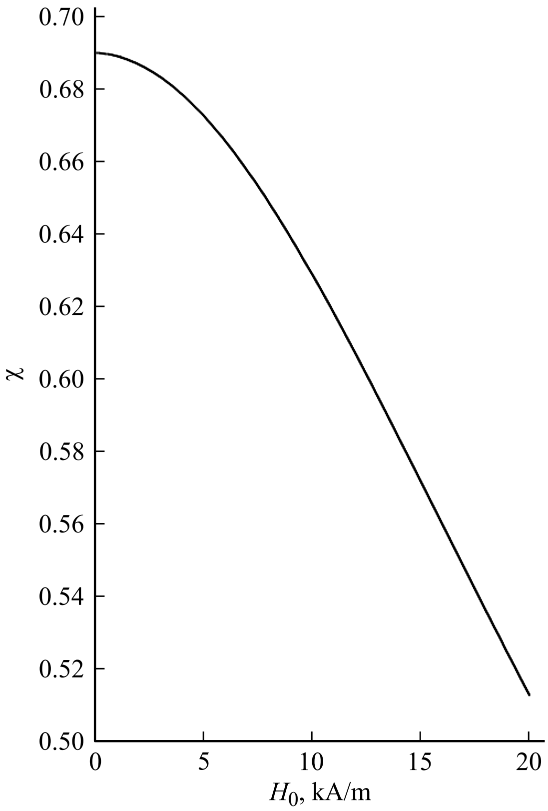

Figure 1. Magnetic susceptibility vs. field strength.

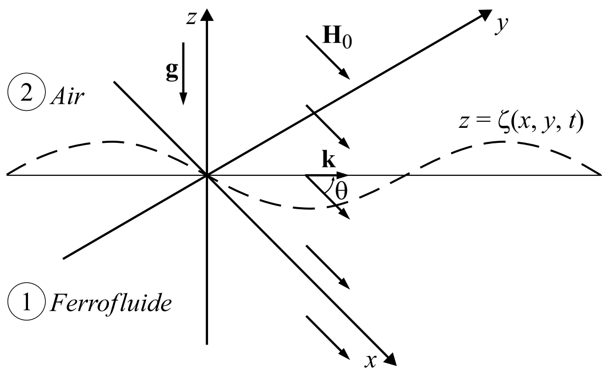

Figure 2. Geometry of problem and designations.

In absence of wave the pressure in ferrofluid is expressed by function $p_0(z) = p_a - \rho g z$, where p_a — atmosphere pressure. Let's elevation of the free surface disturbed by the wave — dashed curve in Fig. 2 — described by equation $z = \xi(x, y, t)$, where t — time.

Magnetic fields perturbed by the change in the shape of the free surface, will be designated as $\mathbf{H}_j(x, y, z, t) = [H_0 + h_{jx}(x, y, z, t)]\mathbf{a}_x + h_{jy}(x, y, z, t)\mathbf{a}_y + h_{jz}(x, y, z, t)\mathbf{a}_z$, $j = 1, 2$, where $h_{jx}(x, y, z, t)$, $h_{jy}(x, y, z, t)$, $h_{jz}(x, y, z, t)$ — small perturbations, and $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ — unit vectors along axes x, y, z. Index $j = 1$ relates to region occupied by ferrofluid, and index $j = 2$ to air above the ferrofluid.

As in studied mediums the electric current are absent, then potentials $\psi_j(x,y,z,t)$ of magnetic field perturbations exist, and we can write $\mathbf{h}_j(x,y,z,t)=\nabla\psi_j$. In this case, $\mathbf{H}_j(x,y,z,t)=\left(H_0+\frac{\partial\psi_j}{\partial x}\right)\mathbf{a}_x+\frac{\partial\psi_j}{\partial y}\mathbf{a}_y+\frac{\partial\psi_j}{\partial z}\mathbf{a}_z$.

Magnetic induction vector in region filled with air is

$$\mathbf{B}_2(x,y,z,t)=\mu_0\left[\left(H_0+\frac{\partial\psi_2}{\partial x}\right)\mathbf{a}_x+\frac{\partial\psi_2}{\partial y}\mathbf{a}_y+\frac{\partial\psi_2}{\partial z}\mathbf{a}_z\right]. \tag{2}$$

Neglecting small values of second order we determine the magnetization vector of ferrofluid $\mathbf{M}(x,y,z,t)=\chi(H_0)\left[\left(H_0+\frac{\partial\psi_1}{\partial x}\right)\mathbf{a}_x+\frac{\partial\psi_1}{\partial y}\mathbf{a}_y+\frac{\partial\psi_1}{\partial z}\mathbf{a}_z\right]$ and magnetic induction vector

$$\mathbf{B}_1(x,y,z,t)=\mu_0[1+\chi(H_0)]\left[\left(H_0+\frac{\partial\psi_1}{\partial x}\right)\mathbf{a}_x+\frac{\partial\psi_1}{\partial y}\mathbf{a}_y+\frac{\partial\psi_1}{\partial z}\mathbf{a}_z\right]. \tag{3}$$

In system of Maxwell equations one of the equations is condition of absence of free magnetic charges [16]. Considering (2), (3) this condition is written as follows:

$$\frac{\partial^2\psi_j}{\partial x^2}+\frac{\partial^2\psi_j}{\partial y^2}+\frac{\partial^2\psi_j}{\partial z^2}=0,\quad j=1,2. \tag{4}$$

On free surface the boundary conditions of magnetostatics are as follows

$$z=0:\quad \psi_1=\psi_2,\ \mu(H_0)\frac{\partial\psi_1}{\partial z}-\mu_0\frac{\partial\psi_2}{\partial z}=\mu_0H_0\chi(H_0)\frac{\partial\xi}{\partial x}. \tag{5}$$

In regions 1, 2 far from the free surface the perturbations of magnetic fields disappear.

The linear system of equations of hydrodynamics is written as follows:

$$\mathrm{div}\mathbf{u}=0,\ \rho\frac{\partial\mathbf{u}}{\partial t}=-\nabla p_1+\rho\mathbf{g}+\mu_0H_0\chi(H_0)\nabla\frac{\partial\psi_1}{\partial x}. \tag{6}$$

Here $\mathbf{u}=\mathbf{u}(x,y,z,t)=(u_x,u_y,u_z)$ — ferrofluid velocity, ρ — density, $p_1=p_1(x,y,z,t)$ — perturbation of pressure caused by wave.

After introduction of velocity potential $\mathbf{u}=\nabla\varphi(x,y,z,t)$ the first equation (6) takes form

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0, \tag{7}$$

and from second equation (6) the linearized Cauchy–Lagrange integral follows

$$p_1(x,y,z,t)=-\rho\frac{\partial\varphi}{\partial t}-\rho gz+\mu_0H_0\chi(H_0)\frac{\partial\psi_1}{\partial x}.$$

On free surface the kinematic and dynamic conditions are written as follows:

$$z=0:\quad \frac{\partial\xi}{\partial t}=\frac{\partial\varphi}{\partial z},$$

$$\rho\frac{\partial\varphi}{\partial t}+\rho gz-\mu_0H_0\chi(H_0)\frac{\partial\psi_1}{\partial_x}-\alpha\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right)=0. \tag{8}$$

Further the linked with each other problems of magnetostatics (4), (5) and hydrodynamics (7), (8) are considered.

2. Anisotropy of wave propagation

Let us introduce designation $\mathbf{r}=x\mathbf{a}_x+y\mathbf{a}_y$. Let's flat at initial time free surface is subjected to small perturbation, when coordinates of points of the surface and desired functions in equations (4), (7) are expressed using normal modes [17] proportional to $\exp[i(\mathbf{kr}-\omega t)]$:

$$(\xi(x,y,t),\psi_j(x,y,z,t),\varphi(x,y,z,t))$$
$$=(Z,\Psi_j(z),\Phi(z))\exp[i(\mathbf{kr}-\omega t)],\quad j=1,2. \tag{9}$$

Here i — imaginary unit, $\mathbf{k}=k_x\mathbf{a}_x+k_y\mathbf{a}_y$ — wave vector, Z — constant, and frequency ω is determined using equations (4), (7) and boundary conditions (5), (8).

Substituting expressions (9) in Laplace equations (4), (7) we obtain

$$\frac{d^2\psi_1}{dz^2}-k^2\Psi_1=0,\quad \frac{d^2\psi_2}{dz^2}-k^2\Psi_2=0,$$

$$\frac{d^2\Phi}{dz^2}-k^2\Phi=0,\quad k^2=k_x^2+k_y^2. \tag{10}$$

Considering (9) the boundary conditions (5), (8) are written as follows:

$$z=0:\ \Psi_1=\Psi_2,\ (1+\chi(H_0))\frac{d\psi_1}{dz}-\frac{d\psi_2}{dz}=ik_xZH_0\chi(H_0),$$

$$i\omega Z+\frac{d\Phi}{dz}=0,\ i\rho\omega\Phi-Z(\rho g+\alpha k^2)+ik_x\mu_0H_0\chi(H_0)\Psi_1=0. \tag{11}$$

In region 1 it is easy to indicate the tending to zero at $z\to-\infty$ solutions of first and third equations (10):

$$\Psi_1=A_1\exp(kz),\quad \Phi=A_2\exp(kz). \tag{12}$$

In region 2 we have tending to zero at $z\to+\infty$ solution of second equation (10):

$$\Psi_2=A_3\exp(-kz). \tag{13}$$

Here A_1, A_2, A_3 are arbitrary constants.

When substituting solutions (12), (13) into boundary conditions (11) we obtain system of linear homogeneous equations relatively to A_1, A_2, A_3, Z:

$$A_1-A_3=0,\ [1+\chi(H_0)]kA_1+kA_3-ik_xZH_0\chi(H_0)=0,$$

$$kA_2+i\omega Z=0,\ ik_x\mu_0H_0\chi(H_0)A_1+i\rho\omega A_2-(\rho g+\alpha k^2)Z=0. \tag{14}$$

Then and only then the system (14) has solutions that differ from zero one, when its determinant is zero. After determinant calculation and equating to zero we obtain the dispersion relation

$$\omega^2=gk+\frac{\mu_0H_0^2\chi(H_0)\cos^2\theta}{\rho[2+\chi(H_0)]}k^2+\frac{\alpha}{\rho}k^3. \tag{15}$$

Then we determine

$$A_1=A_3=\frac{iH_0\chi(H_0)Z\cos\theta}{2+\chi(H_0)},\ A_2=-\frac{i\omega}{k}Z. \tag{16}$$

Using (9), (12), (16) we determine solution of hydrodynamic problem (7), (8):

$$\varphi(x, y, z, t) = \frac{\omega}{k} Zq(x, y, t) \exp(kz),$$

$$\xi(x, y, t) = Z \cos[k(x \cos\theta + y \sin\theta) - \omega t],$$

where $q(x, y, t) = \sin[k(x \cos\theta + y \sin\theta) - \omega t]$.

As in case of waves on calm water [9], the main wave characteristics are phase $c(k) = \omega/k$ and group $C(k) = d\omega/dk$ velocities. Opposite to [9] these characteristics depend also on the parameters H_0 and θ. Considering (15), we determine

$$c(k) = \left[\frac{g}{k} + b(H_0)\cos^2\theta + \frac{\alpha}{\rho}k\right]^{1/2}, \qquad (17)$$

where $b(H_0) = \frac{\mu_0 H_0^2 \chi^2(H_0)}{\rho(2+\chi(H_0))}$,

$$C(k) = 0.5c(k)Q(k)/R(k),$$

where $Q(k) = 1 + \frac{2b(H_0)\cos^2\theta}{g}k + \frac{3\alpha}{\rho g}k^2$,

$R(k) = 1 + \frac{b(H_0)\cos^2\theta}{g}k + \frac{\alpha}{\rho g}k^2$.

At $k = k_m = \sqrt{\rho g/\alpha}$, $\theta \in [0, \pi/2]$ the phase velocity has minimum

$$c_m = c(k_m) = \left(2\sqrt{\frac{\alpha g}{\rho}} + b(H_0)\cos^2\theta\right)^{1/2}.$$

For considered ferrofluid $k_m = 4.69\,\mathrm{cm}^{-1}$. Wave number k_m corresponds to wavelength $\lambda_m = 2\pi/k_m = 1.34$ cm. As in case [9] of common fluid, $c_m = C(k_m)$ at $\theta \in [0, \pi/2]$.

As example we study effect of magnetic field with strength $H_0 = 10$ kA/m on phase and group velocities of wave upon $\theta \in [0, \pi/2]$ increasing.

Fig. 3 in plane (k, c) shows passing through points $(k_m, c(k_m, 0))$, $(k_m, c(k_m, \pi/4))$, $(k_m, c(k_m, \pi/3))$ the function graphs (17). From Figure we see that for any value of $c(k) > c(k_m)$ in each of cases $\theta = 0, \pi/4, \pi/3$ there are two allowable wave numbers.

According to terms used to describe the dispersive waves in common fluid [9], in case of the ferrofluid the region $0 < k < k_m$ is gravitational branch, and region $k > k_m$ — capillary branch.

Fig. 4 shows graphs of function $s(k) = C(k)/c(k)$ plotted for $\theta = 0, \pi/4, \pi/3$. It is evident that $c(k) > C(k)$ in range $0 < k < k_m$, whereas $c(k) < C(k)$ at $k > k_m$.

In paper [4] when deriving the dispersion relation (42) the system of coordinates was used, where axis x is directed opposite to the vector of gravity acceleration $\mathbf{g}$, and axes y, z are horizontal. Regarding wave which length is smaller as compared to thickness of ferrofluid layer interfaced with air the dispersion relation (42) after simplification is

$$\omega^2 = gk + \frac{\alpha}{\rho_b}k^3 + \frac{\mu_0 M^2}{\rho_b}\frac{kk_y^2}{\beta_b(\chi+1)+k},$$

where $k = \sqrt{k_y^2 + k_z^2}$, $\beta_b = \left(\frac{\chi_s+1}{\chi+1}k_y^2 + k_z^2\right)^{1/2}$, $\chi = \frac{M}{H}$, $\chi_s = \frac{dM}{dH}$, ρ_b — ferrofluid density.

From here we find that

$$c(k) = \left[\frac{g}{k} + \frac{\alpha}{\rho_b}k + \frac{\mu_0 H^2\chi^2}{\rho_b}\frac{\cos^2\theta}{1+(1-\kappa\cos^2\theta)^{1/2}}\right]^{1/2},$$

$$\kappa = \frac{\chi - \chi_s}{\chi + 1}.$$

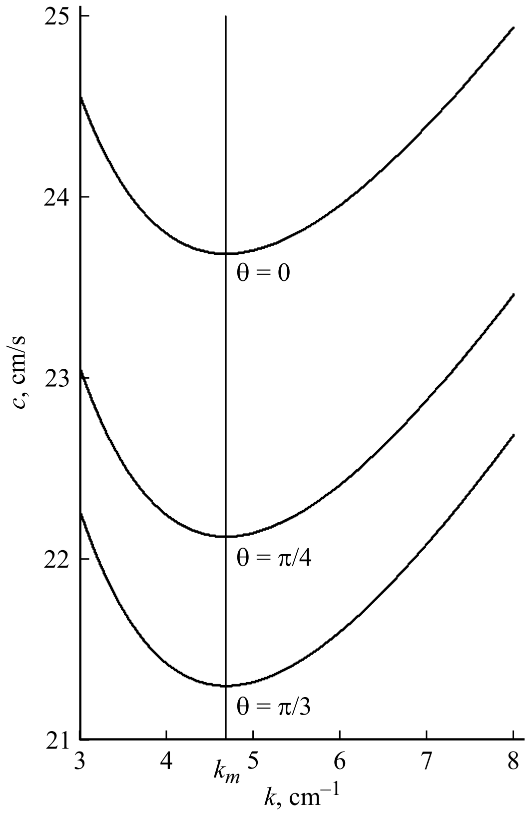

Figure 3. Effect of direction of wave propagation on phase velocity.

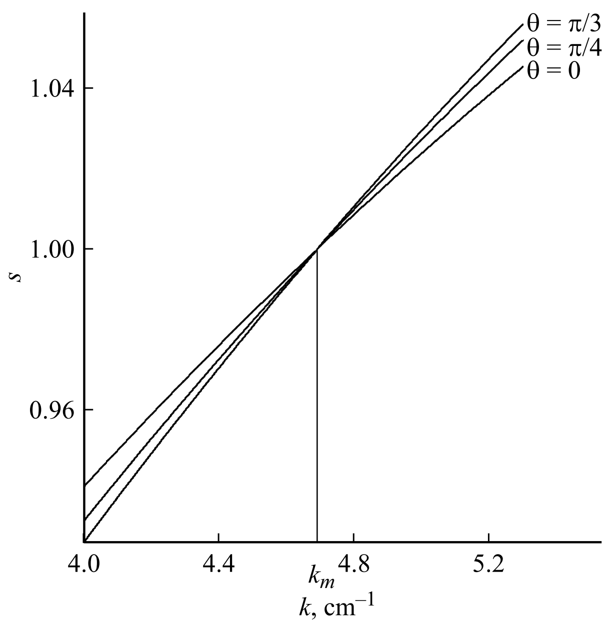

Figure 4. Effect of direction of wave propagation on relation of group velocity to phase velocity.

So, the phase velocity of wave depends on the direction of its propagation in relation to the magnetic field vector.

Conclusion

We discuss the problem of anisotropy of propagation over free surface of ferrofluid of the spatial short wave under action of homogeneous horizontal magnetic field. The anisotropy effect manifests itself in dependence of values of phase and group velocities on $\cos^2\theta$, where θ — angle between the magnetic field vector and wave vector.

On plane of parameters (wave number, phase velocity) the value of wave number k_m, implementing minimum of phase velocity, separates the gravitational branch, where $C(k) < c(k)$, and capillary branch, where $C(k) > c(k)$. In this case, $c(k_m) = C(k_m)$.

If $\theta = \pi/2$, then created by wave change in time of free surface shape does not result in magnetic force occurrence. In this case at fixed value of the wave number the phase and the group velocities in the ferrofluid are equal, respectively, to the phase and the group velocities in common fluid having similar to ferrofluid density and surface tension coefficient.

Conflict of interest

The author declares that he has no conflict of interest.

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