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Correlation analysis of the interaction of plane capillary waves in the regime of developed wave turbulence

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The work presents the results of direct numerical simulation of the turbulence of plane capillary waves propagating along a liquid boundary. The model used is completely nonlinear and takes into account the effects of pumping and dissipation of energy. The calculated turbulence spectrum is in good agreement with the analytical estimate obtained on the basis of the theory of weak wave turbulence under the assumption of the dominant influence of resonant five-wave interactions. The correlation analysis directly demonstrates the presence of nontrivial five-wave interactions of plane capillary waves.

Keywords: capillary turbulence, direct numerical modeling, conformal variables.

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It is known that systems of nonlinear waves may enter a complex chaotic state (wave turbulence regime) as a result of resonant interactions [1]. The interaction conditions for N waves may be written as

$$\omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \ldots \pm \omega(\mathbf{k}_N) = 0, \quad \mathbf{k}_1 \pm \mathbf{k}_2 \ldots \pm \mathbf{k}_N = 0, \tag{1}$$

where wave vector $\mathbf{k}$ and angular frequency ω are bound by dispersion relation $\omega = \omega(\mathbf{k})$. Relations (1) stem from the smallness of nonlinear effects; number N of interacting waves corresponds to nonlinear effects of order $N-1$.

The dispersion relation for capillary waves on the surface of a liquid takes the form

$$\omega = (\sigma/\rho)^{1/2} k^{3/2}, \qquad k = |\mathbf{k}|, \tag{2}$$

where σ and ρ are the surface tension and the mass density of a liquid, respectively. Together with relation (2), conditions (1) form a system of nonlinear algebraic equations. Cascade generation of small-scale harmonics may proceed when this system has nontrivial solutions. This process leads to chaotization of the evolution of waves and the development of wave turbulence. In isotropic three-dimensional geometry, the system of equations (1) and (2) has nontrivial solutions at $N = 3$; i.e., resonant three-wave interactions, which represent the decay of one wave into two ($1 \to 2$), are dominant. Zakharov and Filonenko [2] have obtained the corresponding spectrum of capillary turbulence for the Fourier transform of function $\eta(\mathbf{r}, t)$ that specifies the shape of the liquid surface, $S_\eta(k) = |\eta_k|^2$:

$$S_\eta(k) = C^k_{3w} P^{1/2} (\sigma/\rho)^{-3/4} k^{-15/4}, \tag{3}$$

where C^k_{3w} is a dimensionless constant and P is the rate of energy dissipation per unit surface area. The validity of spectrum (3) of isotropic capillary turbulence on the surface of a liquid has already been verified with high accuracy both experimentally [3,4] and numerically [5–7]. The case of anisotropic surface perturbations with the examined waves propagating in one direction (i.e., collinear waves) is significantly different. In this scenario, conditions of resonant interaction (1) for $N = 3$ and 4 are no longer satisfied (the system has only trivial solutions) [8]. Trivial resonant interactions do not induce the generation of new waves of different scales [1]. The results of experimental study [9] carried out for collinear waves on the surface of mercury indicate that the influence of resonant five-wave resonant interactions $N = 5$ corresponding to the fourth order of nonlinearity is dominant. Wave interaction conditions (1) may always be satisfied for resonances of such a high order. Having performed a dimensional analysis of weak turbulence spectra, the authors of [9] proposed an estimate for the spectrum of capillary wave turbulence in quasi-one-dimensional geometry:

$$S_\eta(k) = C^k_{5w} P^{1/4} (\sigma/\rho)^{-3/8} k^{-27/8}, \tag{4}$$

where C^k_{5w} is the corresponding dimensionless constant.

Turbulence spectrum (4) was reproduced numerically within a strongly nonlinear plane-symmetric model in recent study [10]. It should be noted that in degenerate one-dimensional geometry, so-called coherent structures (solitons or shock waves) may take a dominant role in the development of wave turbulence (see, e.g., [11,12]). Such structures may have a significant effect on the spectrum of observed turbulence. Thus, a convincing demonstration of the dominant influence of resonant five-wave interactions cannot rely on the quantitative agreement of the results reported in [10] with spectrum (4) only: one needs to prove that resonances (1) for $N = 5$ are indeed observed in direct numerical simulations. This was exactly the aim of the present study. It is demonstrated below through the use of high-order correlation functions that the interaction of nonlinear capillary waves is characterized accurately by conditions (1).

Let us discuss briefly the specifics of the computational model of capillary wave turbulence. The motion of a liquid is considered to be plane-symmetric; i.e., the complete physical model is two-dimensional. Let us introduce Cartesian coordinate system $\{x, y\}$ such that equation $y = \eta(x, t)$ specifies the deviation of the free surface from unperturbed state $y = 0$. The motion of a liquid is considered to be potential, and its evolution is characterized by the unsteady Bernoulli equation and the kinematic boundary condition (see details in [10]). The model is based on the conformal transformation of the region occupied by a liquid to the half-plane of new conformal variables $\{u, v\}$; i.e., initially independent coordinates $\{x, y\}$ are now regarded as functions $X(u, v)$ and $Y(u, v)$. The liquid surface corresponds to line $v = 0$: $y = Y(u, t)$, $X = u - \hat{H}Y(u, t)$, and $\psi = \Psi(u, t)$. Here, $\hat{H}$ is the Hilbert transform defined in the Fourier space as $\hat{H} f_k = i\,\mathrm{sgn}(k) f_k$, and ψ is the velocity potential at the liquid boundary. The shape of the liquid boundary is expressed implicitly: $\eta(x, t) = Y[X(u, t), t]$. For brevity, we omit the complete derivation of equations of motion [13] and go straight to the resulting model system:

$$Y_t = \left(Y_u \hat{H} - X_u\right)\frac{\hat{H}\Psi_u}{J} - \hat{\gamma}_k Y, \qquad (5)$$

$$\Psi_t = \frac{(\hat{H}\Psi_u)^2 - \Psi_u^2}{2J} + \hat{H}\left(\frac{\hat{H}\Psi_u}{J}\right)\Psi_u + \frac{X_u Y_{uu} - Y_u X_{uu}}{J^{3/2}} + F(k, t) - \hat{\gamma}_k \Psi, \qquad (6)$$

where $J = X_u^2 + Y_u^2$ is the Jacobian of the conformal transformation, $\hat{\gamma}_k$ is the viscosity operator, and $F(k, t)$ is a random driving force acting on large scales. We switched to dimensionless variables in Eqs. (5) and (6), having set $\sigma = 1$ and $\rho = 1$. Differential and integral operators in Eqs. (5) and (6) are calculated using spectral methods with 8192 Fourier harmonics in total; i.e., the boundary conditions are periodic. The integration in time is performed by the explicit fourth-order Runge–Kutta method with step $dt = 2.5 \cdot 10^{-6}$. All calculations presented here were performed in a periodic region with length $L = 2\pi$. Calculations demonstrate that the system undergoes a transition to the regime of quasi-steady chaotic motion (wave turbulence) fairly rapidly (in a time on the order of 100–200 dimensionless units) under the influence of an external driving force (see details in [10]). The spatial spectrum of turbulence calculated in the steady state is shown in Fig. 1. It can be seen that the spectrum actually assumes a power-law shape with an exponent close to $-27/8$; i.e., the spectrum is close to estimate (4). The shape of the liquid boundary behaves in a rather complex irregular way (see the inset in Fig. 1).

Let us apply correlation functions, which are used widely in statistical physics [14], in the analysis of wave interactions.

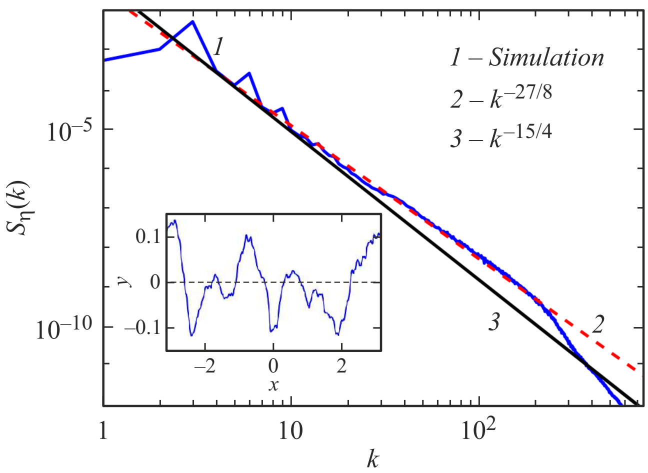

Figure 1. Spatial spectrum of turbulence calculated in the steady state (curve *1*). Line *2* corresponds to spectrum (4); line *3*, to Zakharov–Filonenko spectrum (3). The inset shows the calculated surface shape in the quasi-steady chaotic state (wave turbulence) at a certain point in time.

To analyze the three-wave interaction, we construct third-order correlator $B(k_1, k_2)$ (so-called bicoherence):

$$B(k_1, k_2) = \frac{|\langle \eta_{k_1}\eta_{k_2}\eta^*_{k_1+k_2}\rangle|}{\left[\langle|\eta_{k_1}\eta_{k_2}|^2\rangle\langle|\eta_{k_1+k_2}|^2\rangle\right]^{1/2}}, \qquad (7)$$

where η_k is the spatial Fourier transform of function $\eta(x, t)$ calculated at time instant t, $\langle f\rangle$ denotes averaging of function f over time, and the asterisk denotes complex conjugation. The denominator in formula (7) is chosen in such a way that the value of $B(k_1, k_2)$ varies from 0 (no correlation) to 1 (complete correlation). At $N = 3$, system (1) has trivial solutions only: $k_1 = 0$ and $k_2 = 0$. Figure 2, *a* presents bicoherence (7) calculated based on direct numerical simulation data (time averaging was performed within a narrow time interval $\Delta t = 0.1$). It can be seen that correlations are found only in the vicinity of the coordinate axes corresponding to trivial solutions. These correlations arise as a result of nonlinear frequency broadening when Eqs. (1) turn into an inequality within a certain region δ: $\omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \ldots \pm \omega(\mathbf{k}_N) \leqslant \delta$, and the interaction of waves is called quasi-resonant. This area is bounded by white solid lines in Fig. 2, *a*. Thus, calculated correlation function (7) does indeed demonstrate that the system features only trivial three-wave interactions and a narrow region of quasi-resonances in which short waves interact with long ones. These quasi-resonances do not enable local transfer of energy from large scales to small ones.

The analysis of wave resonances of the next (fourth) order is based on a fourth-order correlator (tricoherence)

$$T(k_1, k_2, k_3) = \frac{|\langle \eta_{k_1}\eta_{k_2}\eta^*_{k_3}\eta^*_{k_1+k_2-k_3}\rangle|}{\left[\langle|\eta_{k_1}\eta_{k_2}|^2\rangle\langle|\eta_{k_3}\eta_{k_1+k_2-k_3}|^2\rangle\right]^{1/2}}. \qquad (8)$$

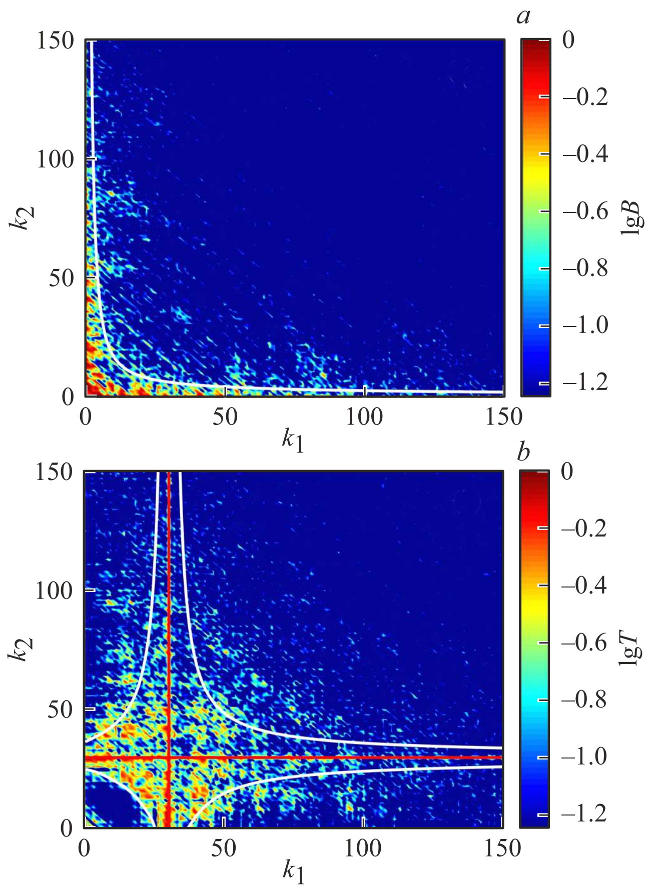

Figure 2. Calculated correlation functions of the third (a) and fourth (b) order on a logarithmic scale. Red solid lines correspond to trivial solutions of system (1), and white solid lines are the boundaries of regions of quasi-resonances. A color version of the figure is provided in the online version of the paper.

Correlator (8) characterizes the probability of wave scattering: $2 \to 2$. For convenience of graphical representation of tricoherence, we fix one of the wave vectors: $k_3 = 30$. Calculated correlator (8) is shown in Fig. 2, b. Trivial solutions $k_1 = k_3$ and $k_2 = k_3$ and a fairly wide region of quasi-resonant interaction are seen clearly in the plot. No nontrivial solutions are observed. Thus, correlators (7) and (8) are in good agreement with conditions (1) for $N = 3$ and 4.

Next, we consider a fifth-order correlator (quadro-coherence)

$$Q(k_1, k_2, k_3, k_4) = \frac{|\langle \eta_{k_1}\eta_{k_2}\eta_{k_3}\eta^*_{k_4}\eta^*_{k_1+k_2+k_3-k_4}\rangle|}{\left[\langle|\eta_{k_1}\eta_{k_2}\eta_{k_3}|^2\rangle\langle|\eta_{k_4}\eta_{k_1+k_2+k_3-k_4}|^2\rangle\right]^{1/2}}, \tag{9}$$

which characterizes $3 \to 2$ interactions. To demonstrate correlator (9), we fix two wave numbers: $k_3 = 15$ and $k_4 = 40$. The result of calculation of quadro-coherence is presented in Fig. 3. This pattern differs fundamentally from the one shown in Fig. 2. In the case of $N = 5$, system (1) does indeed have an exact nontrivial solution, which is denoted by the solid red line in Fig. 3. The correlations found using formula (9) are localized in a narrow region near the nontrivial solution of system (1). Note that the obtained result agrees closely with the experimental data from [9], where the possibility of five-wave resonant interactions has also been demonstrated.

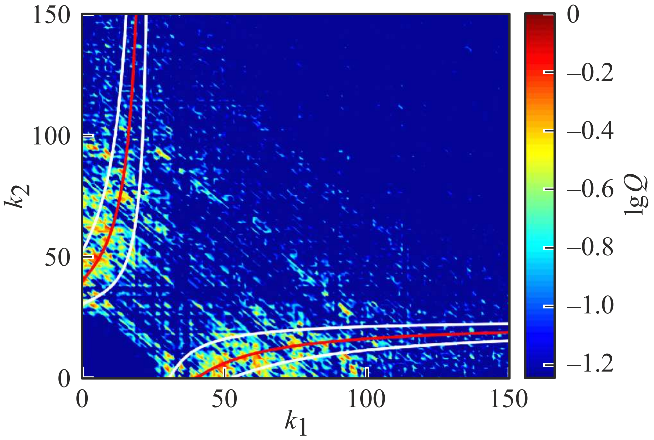

Figure 3. Calculated fifth-order correlation function (9) on a logarithmic scale. Red solid lines correspond to nontrivial solutions of system (1), and white solid lines are the boundaries of the region of quasi-resonances. A color version of the figure is provided in the online version of the paper.

Thus, correlations of plane capillary waves in the regime of developed wave turbulence were analyzed based on direct numerical modeling data. The results of modeling did not only reveal a good agreement of the calculated turbulence spectrum with the theory of weak turbulence, but also demonstrated directly the feasibility of implementation of nontrivial five-wave interactions of plane-symmetric capillary waves.

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Conflict of interest

The authors declare that they have no conflict of interest.

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