

03
Unidirectional pulses: relatively undistorted quasi-spherical waves, Fourier-Bessel integrals, and plane-waves decompositions

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Received February 08, 2024

Revised April 06, 2024

Accepted April 06, 2024

A theoretical description of a class of unidirectional axisymmetric localized pulses, is given. The equivalence of their representations in the form of relatively undistorted quasi-spherical waves, in the form of Fourier-Bessel integrals and in the form of a superposition of plane waves with wave vectors having positive projections on a given direction is established.

Keywords: localized wave packets, unidirectional pulses, closed-form solutions.

DOI: 10.61011/EOS.2024.04.58884.6006-24

1. Introduction

In recent years, there has been an increased interest in localized solutions of the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

(with $x, y,$ and z Cartesian coordinates, t time, and $c > 0$ wave speed assumed constant), having the property of unidirectionality [1–10]. Real and imaginary parts of such solutions can be used as components of the Hertz’s vector in the construction of unidirectional electromagnetic pulses.

One of the formulations of unidirectionality [7] consists in the requirement that only homogeneous plane waves traveling in directions forming an angle with a certain chosen direction not exceeding $\frac{\pi}{2}$, are present in the decomposition of the solution in plane waves. This property expresses the requirement, natural from a physical point of view, that the mathematical model of the pulse describe its propagation strictly from the source. Unidirectional pulses are sometimes called causal [3,4]. It is noteworthy that the unidirectionality understood in the above sense does not exclude the possibility that in some spatiotemporal regions the projection of the energy flow vector on the chosen direction may turn out to be negative [8,9,11].

In what follows, the chosen direction of propagation will be the direction of the z axis. Accordingly, we will call the solution unidirectional if the z projections of the wave vectors of its plane-wave constituents are non-negative. Trivial examples of unidirectional solutions are a plane wave and a finite combination of plane waves that are not spatially localized. In this paper, we address solely unidirectional

pulses, localized with respect to all spacial coordinates at any fixed instant in time.

The first results on the construction of unidirectional pulses were based on the consideration of axisymmetric solutions of the equation (1) in the form of Fourier-Bessel integrals:

$$u = u(\rho, z, t) = \int_0^\infty d\omega e^{i\omega t} \int_0^{\omega/c} dk_z A(k_z, \omega) e^{-ik_z z} \times J_0(\rho \sqrt{\omega^2/c^2 - k_z^2}), \tag{2}$$

where $\rho = \sqrt{x^2 + y^2}$, with fairly arbitrary weight functions A [1–3]. A proper choice of such a weight allowed to find several solutions expressed in terms of elementary functions. The simplest localized unidirectional solution, however, was found differently and was based on a lucky trick [5,6], which used a partial fraction decomposition of the well-known splash pulse [12–14]. This solution is

$$u = \frac{1}{S(S - z_*)}, \tag{3}$$

where

$$S = S(t, \mathbf{R}) = \sqrt{c^2 t_*^2 - \rho^2}, \tag{4}$$

with \mathbf{R} denoting the position vector of the observation point, and

$$z_* = z + i\xi, \quad t_* = t + i\tau. \tag{5}$$

Here, ξ and $\tau > 0$ are free parameters which are assumed real. The square root branch in (4) is chosen so that $S|_{x=y=0} = ct_*$, in which case for arbitrary values of t, \mathbf{R}

the inequality $\Im S \geq c\tau$ [6] holds. The solution (3) is non-singular under the condition $\xi < c\tau$, and then its energy is finite [6,7]. We also note that with proper choice of free parameters ξ and τ , this solution can model pancake-shaped, ball-shaped and needle-shaped focused pulses. The proof of its unidirectionality was presented in [7,9].

On the other hand, Besieris and Saari [9] (see also [2,10]) noted that a special class of relatively undistorted waves is important in the description of unidirectional wave propagation. This is the name of the solutions of the equation (1) of the form

$$u = gf(\theta), \tag{6}$$

where the amplitude $g = g(x, y, z, t)$ and the phase $\theta = \theta(x, y, z, t)$ functions are fixed, and waveform f is an arbitrary function [15,16]. If the phase function θ is complex-valued, then the waveform f must be analytical in the range of values θ [17].

The class of axisymmetric solutions we are interested in has the form

$$u = \frac{f(\theta)}{S}, \tag{7}$$

where

$$\theta = S - z - ib, \tag{8}$$

$b = c\tau > 0$, with an arbitrary waveform f analytical in the upper half-plane \mathbb{C}^+ [6]. The imaginary shift ib in (8) is introduced in order to have the upper half-plane as the range of values of the phase θ .

In order for solutions of the form (7) to describe a localized wave, we will require a fairly fast decrease (no slower than $1/|\theta|$) of the function $|f(\theta)|$ when $|\theta| \rightarrow \infty$.

Let us explain how one can easily come to such solutions. Let us consider a relatively undistorted wave corresponding to spherical waves [15],

$$u = \frac{f(R - ct)}{R}, \tag{9}$$

$R = \sqrt{x^2 + y^2 + z^2}$. The equation (1) is invariant under replacement

$$z \mapsto i(ct + ib), \quad ct \mapsto i(z + ib). \tag{10}$$

where b is a free real constant. To match with (3)–(5), we take $b = c\tau > 0$. Under the transformation (10), the phase function $R - ct$ in (9) goes into the expression (8), which we will call it quasi-spherical phase. The imaginary part of (8) is non-negative for all values of spatial and temporal variables. Redefining the waveform using the rule $f(i\theta)/i \mapsto f(\theta)$, we obtain a class of relatively undistorted waves of the form (7), which we will call quasi-spherical.

Note that the expression (3) is a special case of (7) when $f(\theta) = \frac{1}{\theta + i(b - \xi)}$. A number of more complex, although expressed through elementary functions, solutions from this class were found in [1,3,10]. A solution close to that found in [5,6], but less general — with $\xi = 0$ — is presented in [8]. The article [9] provides an overview of such solutions.

An example of a unidirectional solution for a time-harmonic regime is given in [18]. This solution has an asymptotic behavior corresponding to a Gaussian beam with arbitrary astigmatism.

In this note, we establish a relation between solutions described by (7) and (2). Further, we represent these solutions in the form of a superposition of plane waves. Our approach is based on a technique for studying localized solutions that goes back to Blagoveshchenskii [19] and Moses–Prosser [20]. It rests upon formulas expressing the solution through its asymptotic behavior in the far zone at large time. This technique turns out to be convenient, in particular, for calculating such characteristics of a localized pulse as energy, momentum and orbital angular momentum [7,21].

2. The Blagoveshchenskii — Moses — Prosser approach and the unidirectionality of quasi-spherical waves

2.1. Blagoveshchenskii — Moses — Prosser approach

We denote the position vector by $\mathbf{R} = xi + yj + zk$, where i, j and k are unit vectors along the coordinate axes. Let $\mathbf{n} = \frac{\mathbf{R}}{R}$, $|\mathbf{n}| = 1$ be the related to \mathbf{R} unit vector, and $R = |\mathbf{R}| = \sqrt{x^2 + y^2 + z^2}$ be the distance to the Cartesian origin.

Consider an arbitrary smooth localized solution of the wave equation (1), assuming that R and ct grow consistently, i.e., that their difference

$$s = R - ct \tag{11}$$

remains constant. Evidently, $\mathbf{R} = (ct + s)\mathbf{n}$.

In the works of Blagoveshchenskii [19] and Moses–Prosser [20], it was found that for any solution of the wave equation decreasing rapidly enough at $R \rightarrow \infty$, for any fixed s and any direction \mathbf{n} there exists the limit

$$F(s, \mathbf{n}) = \lim_{t \rightarrow \infty} [ct u(t, (ct + s)\mathbf{n})]. \tag{12}$$

The limit (12) characterizes, at large values of time, the amplitude of the pulse in the direction of \mathbf{n} . For a unidirectional (along the z axis) wave packet, obviously $F(s, \mathbf{n}) \equiv 0$ for all \mathbf{n} whose projections on the z axis are negative, i.e. $\mathbf{n} \cdot \mathbf{k} < 0$, where $\mathbf{n} \cdot \mathbf{k}$ is the scalar product of the vectors \mathbf{n} and \mathbf{k} .

We will characterize the direction \mathbf{n} by the angles χ and φ of the spherical coordinate system with the polar axis z :

$$\mathbf{n} = \sin \chi \cos \varphi \mathbf{i} + \sin \chi \sin \varphi \mathbf{j} + \cos \chi \mathbf{k},$$

$0 \leq \varphi < 2\pi$, $0 \leq \chi \leq \pi$. The unidirectionality condition takes the form

$$F(s, \mathbf{n}) \equiv 0, \quad \frac{\pi}{2} < \chi \leq \pi. \tag{13}$$

The nontrivial result of Blagoveshchenskii – Moses – Prosser (see [16,19,20]) is that the solution of u at any point of \mathbf{R} at any moment t is representable through the limit (12) as follows:

$$u(t, \mathbf{R}) = \frac{1}{2\pi} \iint_{|\mathbf{N}|=1} F'(\mathbf{N} \cdot \mathbf{R} - ct, \mathbf{N}) d^2\mathbf{N}, \quad (14)$$

where the notation is introduced

$$F'(s, \mathbf{N}) = \frac{\partial F(s, \mathbf{N})}{\partial s}.$$

The integration is carried out over the unit sphere $|\mathbf{N}| = 1$, and $d^2\mathbf{N}$ denotes the element of its surface area. In spherical coordinates $\mathbf{N} = \sin \mathcal{X} \cos \phi \mathbf{i} + \sin \mathcal{X} \sin \phi \mathbf{j} + \cos \mathcal{X} \mathbf{k}$, and the area element of the sphere takes the form $d^2\mathbf{N} = \sin \mathcal{X} d\mathcal{X} d\phi$.

The formula (14) represents the solution u in the form of a superposition of nonstationary plane waves.

2.2. Quasi-spherical wave (7) at large time and large distance

Let us find the limit (12) for the solution (7). If $\cos \chi \neq 0$, then at $t \rightarrow +\infty$

$$\begin{aligned} S = S(t, (ct + s)\mathbf{n}) &= \sqrt{(ct + ib)^2 - (ct + s)^2 \sin^2 \chi} \\ &\approx ct |\cos \chi| + \frac{ib - s \sin^2 \chi}{|\cos \chi|} \approx ct |\cos \chi|, \end{aligned}$$

so that

$$\begin{aligned} \theta = S - (ct + s) \cos \chi - ib &\approx ct (|\cos \chi| - \cos \chi) \\ &+ \frac{-s(\sin^2 \chi + \cos \chi |\cos \chi|) + ib(1 - |\cos \chi|)}{|\cos \chi|}. \end{aligned}$$

For directions \mathbf{n} making an obtuse angle with the z axis, $\cos \chi < 0$, that is, $\chi > \frac{\pi}{2}$, we have $u \approx f(2ct |\cos \chi|) / (ct |\cos \chi|)$, and since $f(2ct |\cos \chi|) \rightarrow 0$, from (12) it follows that

$$F(s, \mathbf{n}) = 0.$$

For directions \mathbf{n} making an acute angle with the z axis, $\cos \chi > 0$,

$$u(t, (ct + s)\mathbf{n}) \approx \frac{1}{ct \cos \chi} f\left(\frac{-s + ib(1 - \cos \chi)}{\cos \chi}\right),$$

and (7) implies

$$F(s, \mathbf{n}) = \frac{1}{\cos \chi} f\left(\frac{-s + ib(1 - \cos \chi)}{\cos \chi}\right).$$

Finally, for $\chi = \frac{\pi}{2}$ a similar calculation gives the value

$$\begin{aligned} F(s, \mathbf{n}) &= \lim_{t \rightarrow \infty} \left[\sqrt{\frac{ct}{2(ib - s)}} f\left(\sqrt{2ct(ib - s)}\right) \right] \\ &= \frac{1}{2(ib - s)} \lim_{\theta \rightarrow \infty} \theta f(\theta). \end{aligned}$$

This limit is finite if $|f(\theta)|$ decreases not slower than $|\theta|^{-1}$, as we have assumed. Since the circle $\chi = \frac{\pi}{2}$ does not contribute to the integral (14), the value of $F(s, \mathbf{n})|_{\chi=\frac{\pi}{2}}$ can be replaced by zero and the result written in the form

$$F(s, \mathbf{n}) = \frac{H(\cos \chi)}{\cos \chi} f\left(\frac{-s + ib(1 - \cos \chi)}{\cos \chi}\right), \quad (15)$$

with H the Heaviside step function

$$H(p) = \begin{cases} 1, & p > 0, \\ 0, & p < 0. \end{cases} \quad (16)$$

Thus, since for a quasi-spherical wave in the formula (14) the integration occurs over the forward hemisphere $\mathbf{N} \cdot \mathbf{k} > 0$, the unidirectionality of the pulse (7) is established.

2.3. On angular divergence of quasi-spherical waves

It should be noted that quasi-spherical solutions can describe pulses having not only a significant (as in the examples discussed in [5,6,8]), but also a small angular divergence. Strong angular localization requires a rapid decrease in the modulus of the function $f(\theta)$ with the growth of $\Im\theta$. This property is possessed, for example, by the waveform introduced by Lekner [10], having the form

$$f(\theta) = \exp(iK\theta) / (\theta + ib),$$

(K is a real constant), for which the angular localization in the angle χ has Gaussian character.

A number of examples of a waveform that provides Gaussian localization not only in angles, but also in a longitudinal variable can be found in the work of Kiselev and Perel [22] (see also [16,23]), devoted to wave packets of a different nature.

3. Integral representations of the quasi-spherical wave

3.1. Representation by superposition of non-stationary plane waves

Differentiating the function (15) with respect to the first argument, we obtain

$$F'(s, \mathbf{n}) = -\frac{H(\cos \chi)}{\cos^2 \chi} f'\left(\frac{-s + ib(1 - \cos \chi)}{\cos \chi}\right), \quad \chi \neq \frac{\pi}{2}. \quad (17)$$

Substituting (17) into (14) and replacing \mathbf{n} with \mathbf{N} gives

$$\begin{aligned} u &= -\frac{1}{2\pi} \iint_{\Sigma_+} \frac{d^2\mathbf{N}}{\cos^2 \mathcal{X}} f' \left(\frac{ct - \mathbf{NR} + ib(1 - \cos \mathcal{X})}{\cos \mathcal{X}} \right) \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_0^{\frac{\pi}{2}} \frac{\sin \mathcal{X} d\mathcal{X}}{\cos^2 \mathcal{X}} \times \\ & f' \left(\frac{(ct+ib) - (z+ib) \cos \mathcal{X} - (x \cos \phi + y \sin \phi) \sin \mathcal{X}}{\cos \mathcal{X}} \right), \end{aligned} \quad (18)$$

where Σ_+ denotes the forward semi-sphere $\{|\mathbf{N}| = 1, \mathcal{X} < \frac{\pi}{2}\}$.

3.2. Representation by a superposition of monochromatic plane waves

Let us represent the waveform as follow

$$f(\theta) = \int_0^\infty \hat{f}(\kappa) \exp(i\kappa\theta) d\kappa,$$

this representation holds for fairly rapidly decreasing functions f . Integration is carried out along the positive semi-axis in view of the analyticity of f in the upper half-plane. Then

$$f'(\theta) = i \int_0^\infty \hat{f}(\kappa) \exp(i\kappa\theta) \kappa d\kappa.$$

Therefore,

$$\begin{aligned} u &= -\frac{i}{2\pi} \int_{-\pi}^{\pi} d\phi \int_0^{\frac{\pi}{2}} \frac{\sin \mathcal{X} d\mathcal{X}}{\cos^2 \mathcal{X}} \int_0^\infty \hat{f}(\kappa) \exp \left(i\kappa \times \right. \\ & \left. \frac{(ct+ib) - (z+ib) \cos \mathcal{X} - (x \cos \phi + y \sin \phi) \sin \mathcal{X}}{\cos \mathcal{X}} \right) \\ & \times \kappa d\kappa = -\frac{ia}{2\pi} \int_{-\pi}^{\pi} d\phi \int_0^{\frac{\pi}{2}} \sin \mathcal{X} d\mathcal{X} \int_0^\infty \hat{f}(k \cos \mathcal{X}) \\ & \times \exp [ik((ct+ib) - (z+ib) \cos \mathcal{X} \\ & - (x \cos \phi + y \sin \phi) \sin \mathcal{X})] k dk, \end{aligned} \quad (19)$$

where the replacement $\kappa = k \cos \mathcal{X}$ was made. The right-hand side of (19) can be understood as a volume integral presented in spherical coordinates (k, \mathcal{X}, ϕ) , with $k^2 dk \sin \mathcal{X} d\mathcal{X} d\phi$ being the volume element and the integrand being

$$\hat{f}(k \cos \mathcal{X}) \frac{e^{ik[(ct+ib) - (z+ib) \cos \mathcal{X} - (x \cos \phi + y \sin \phi) \sin \mathcal{X}]}}{k}.$$

The area of integration is the half-space $0 \leq \mathcal{X} < \frac{\pi}{2}$. Passing in (19) to Cartesian coordinates

$$k_z = k \cos \mathcal{X}, \quad k_x = k \sin \mathcal{X} \cos \phi, \quad k_y = k \sin \mathcal{X} \sin \phi,$$

we find

$$\begin{aligned} u &= -\frac{i}{2\pi} \int_0^\infty \hat{f}(k_z) dk_z \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y \\ & \times \frac{e^{i[k(ct+ib) - k_z(z+ib) - k_x x - k_y y]}}{k}, \end{aligned} \quad (20)$$

with $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

Thus, we presented the quasi-spherical wave (7) in the form of expansion in monochromatic plane waves.

Continuing the integrand with zero to negative values of k_z , we obtain

$$u = -\frac{i}{2\pi} \iiint_{\mathbb{R}^3} H(k_z) \hat{f}(k_z) \frac{e^{i[k(ct+ib) - k_z(z+i\xi) - k_x x - k_y y]}}{k} d^3\mathbf{k}, \quad (21)$$

with $d^3\mathbf{k} = dk_x dk_y dk_z$ standing for volume element.

3.3. Representation by a superposition of monochromatic cylindrical waves by the Fourier-Bessel integral

Now we pass in (19) to cylindrical coordinates, $x = \rho \cos \phi$, $y = \rho \sin \phi$, $x \cos \phi + y \sin \phi = \rho \cos(\phi - \phi)$. Integration with respect to ϕ and application of the well-known expression for the Bessel function J_0 , $J_0(m) = \frac{1}{2\pi} \int_0^{2\pi} e^{im \cos \mu} d\mu$, see [24], provides

$$\begin{aligned} u &= -i \int_0^{\frac{\pi}{2}} \sin \mathcal{X} d\mathcal{X} \int_0^\infty \hat{f}(k \cos \mathcal{X}) J_0(k\rho \sin \mathcal{X}) \\ & \times e^{ik[(ct+ib) - (z+ib) \cos \mathcal{X}]} k dk. \end{aligned}$$

By substitution $k_z = k \cos \mathcal{X}$ we present (7) as follows

$$u = -i \int_0^\infty e^{ik(ct+ib)} dk \int_0^k \hat{f}(k_z) J_0(\sqrt{k^2 - k_z^2} \rho) e^{-ik_z(z+ib)} dk_z. \quad (22)$$

Introducing the variable $\omega = ck$ we come up with

$$\begin{aligned} u &= -\frac{i}{c} \int_0^\infty e^{i\omega(t+ib/c)} d\omega \int_0^{\omega/c} \hat{f}(k_z) J_0(\sqrt{(\omega/c)^2 - k_z^2} \rho) \\ & \times e^{-ik_z(z+ib)} dk_z. \end{aligned} \quad (23)$$

which allows the following relation between the waveform $f(\theta)$ in (7) and the weight $A(k_z, \omega)$ in (2):

$$A(k_z, \omega) = -\frac{i}{c} e^{-(\omega/c - k_z)b} \hat{f}(k_z). \quad (24)$$

4. Conclusions

Thus, we have established relationships between several representations of localized unidirectional waves. These are quasi-spherical waves (7), Fourier-Bessel integrals (2) and superpositions of monochromatic (21) and non-stationary (18) plane waves.

Acknowledgments

The authors are grateful to I.Besieris, P.Saari, and J.Lekner for a useful exchange of views, and to N.N.Rosanov and I.A.So for valuable comments.

Conflict of interest

The authors declare that they have no conflict of interest.

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