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Lyon's integral: turbulent thermal conductivity and thickness of the thermal sublayer

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The heat-transfer coefficient was calculated from the Lyonś integral using the two-layer model (thermal sublayer and turbulent flow core). To estimate the turbulent thermal conductivity, the Prandtl mixing length model was used. Under the accepted assumptions, the Lyonś integral had a rather simple form. Numerical solution of the integral yielded the distribution of the heat transfer coefficient over parameter $\mathrm{Re}\sqrt{\mathrm{Pr}}$, which agrees well with the Dittus–Boelter correlation for turbulent heat transfer.

Keywords: Lyonś integral, turbulent thermal conductivity, thermal sublayer, mixing length.

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Being one of the special cases of the energy equation under certain conditions, the Lyonś integral finds application in estimating the heat transfer coefficient in various flows [1,2]. This integral is mainly considered in the form of

$$\frac{1}{\mathrm{Nu}} = 2\int_0^1 \left(\int_0^R URdR\right)^2 \Bigg/ \left[\left(1+\frac{\mathrm{Pr}}{\mathrm{Pr}_t}\frac{\nu_t}{\nu}\right)R\right] dR,$$

where $\mathrm{Nu} = 2\alpha r_0/\lambda$ is the Nusselt number (α is the heat transfer coefficient), $R = r/r_0$ is the dimensionless radius (r_0 is the channel radius), $U = u/U_m$ is the dimensionless longitudinal velocity (U_m is the average flow rate), Pr and Pr_t are the Prandtl numbers (molecular and turbulent ones), ν and ν_t are the kinematic viscosities (molecular and turbulent ones).

In this relationship, the turbulent Prandtl number is believed to interrelate the processes of momentum and heat transfer. However, determination of this parameter is a rather difficult task; attempts on its modeling and even experimental measurement gave cumbersome expressions and ambiguous results [3]. The integration itself is proposed to be performed over three layers (viscous sublayer, transition region and flow core). In some cases, it is believed possible to restrict the task to a two-layer model (viscous sublayer and flow core).

This work is devoted to determining the heat transfer coefficient for a turbulent flow in a circular pipe based on a „simpler" (original) form of the Lyonś integral:

$$\frac{1}{\mathrm{Nu}} = 2\int_0^1 \left(\int_0^R URdR\right)^2 \Bigg/ \left[\left(1+\frac{\lambda_t}{\lambda}\right)R\right] dR,$$

where λ is the molecular thermal conductivity of the working fluid, $\lambda_t = c_p\rho\langle v't'\rangle/(dt/dy)$ is the turbulent thermal conductivity of the flow (angle brackets indicate averaging), y is the transverse coordinate (from the wall), v' are the transverse velocity oscillations, t and t' are the flow temperature and its oscillations.

In addition to the conditions under which the Lyonś integral was obtained, the following assumptions will be made for solving this problem.

1. A two-layer flow model is considered: thermal sublayer (sublayer with molecular thermal conductivity) and flow core.

2. For the flow core, it is assumed that the turbulent velocity profile obeys the „1/7"law.

3. The Prandtl mixing length model is assumed to be valid for both the hydrodynamic and thermal flow disturbances. Their ratio is $l_m/l_{mT} \approx \sqrt{\mathrm{Pr}}$ (similarly to that between thicknesses of developing boundary layers $\delta/\delta_\mathrm{T} \approx \sqrt{\mathrm{Pr}}$).

4. Oscillations of velocity v' and temperature t' strictly correlate with each other (with correlation coefficient $r_{vt} \approx 1$). Turbulence is isotropic. Note that liquid metals ($\mathrm{Pr} \ll 1$) are not considered here.

5. The thermal sublayer thickness is $y1 \approx \delta 1/\mathrm{Pr}^{1/3}$ ($\delta 1$ is the viscous sublayer thickness [4].

6. Friction on the wall is defined by the Blasius formula.

Thus, the Lyonś integral takes the following form:

$$\frac{1}{\mathrm{Nu}} = 2\int_0^{R1} \left(\int_0^R URdR\right)^2 \Bigg/ \left[\left(1+\frac{\lambda_t}{\lambda}\right)R\right] dR$$
$$+\,2\int_{R1}^1 \left(\int_0^R URdR\right)^2 \Bigg/ R\cdot dR.$$

Here we take into account that the turbulent thermal conductivity in the thermal sublayer ($R1 \leqslant R \leqslant 1$) is $\lambda_t = 0$. The higher is λ_t, the higher is heat transfer coefficient Nu.

According to assumption (2), $u/U_0 = (y/r_0)^{1/7}$, where the maximum channel-axis velocity is $U_0 = 1.22U_m$. In the accepted frame of reference, $Y = y/r_0 = 1 - R$. Then the integral in the numerator (denoted as I) will be

$$I = \int_0^R URdR = \int_0^R 1.22(1-R)^{1/7}RdR$$

$$= 1.22\left[\frac{7(1-R)^{15/7}}{15} - \frac{7(1-R)^{8/7}}{8} - \frac{7}{15} + \frac{7}{8}\right].$$

The obtained integral may be approximated by relation $I = 0.52R^2$ over the entire range $0 \leqslant R \leqslant 1$ (with the near-wall deviations of $\pm 5\%$). Then

$$\frac{1}{\mathrm{Nu}} = 2\int_0^1 0.52^2 R^4 / \left[\left(1 + \frac{\lambda_t}{\lambda}\right)R\right] dR$$

$$= 0.54\int_0^{R1} R^3 \Bigg/ \left(1 + \frac{\lambda_t}{\lambda}\right) dR + 0.54\int_{R1}^1 R^3 dR.$$

Consider the denominator (denoted as Z) in the integrand

$$Z = 1 + \frac{\lambda_t}{\lambda} = 1 + c_p\rho\frac{\langle v't'\rangle}{\lambda \partial t/\partial y}.$$

Here all the parameters are presented in the dimensional form; they may be converted into the dimensionless form using characteristic parameters $V' = v'/U_0$, $T' = t'/(T_f - T_w) = t'/\Delta T$, $Y = y/r_0$. Then

$$\langle v'/t'\rangle = U_0 \Delta T \langle V'T'\rangle, \quad dt/dy = \Delta T/r_0 dT/dY.$$

Hence,

$$Z = 1 + c_p\rho\frac{U_0 r_0 \langle V'T'\rangle}{\lambda \partial T/\partial Y}.$$

If

$$\mathrm{Re} = \rho U_m d/\mu = 2\rho U_0 r_0/(1.22\mu)$$

obtain

$$Z = 1 + 0.61\mathrm{RePr}\frac{\langle V'T'\rangle}{\partial T/\partial Y}.$$

As per assumption (3), in the case of isotropic turbulence $v' \sim u' = l_m du/dy$ (($l_m = 0.4y$ is the mixing length). Let us present temperature oscillations similarly to the flow oscillations: $t' = l_{mT} dt/dy$ ((l_{mT} is the mixing length for temperature disturbances), where $l_{mT} \approx l_m/\sqrt{\mathrm{Pr}}$. Then, if assumption (4) is taken unto account, the set of parameters comprised in the turbulent thermal conductivity definition will be

$$\frac{\langle V'T'\rangle}{\partial T/\partial Y} = \frac{0.4Y(dU/dY)0.4Y(dT/dY)}{\sqrt{\mathrm{Pr}}(dT/dY)} = \frac{0.16Y^2(dU/dY)}{\sqrt{\mathrm{Pr}}}.$$

Note that assumption (4) seems possible if there are no oscillations of the wall temperature or heat flow q_w independent of flow disturbances. Then

$$Z = 1 + \frac{\lambda_t}{\lambda} = 1 + 0.61\mathrm{Re}\sqrt{\mathrm{Pr}} \cdot 0.16Y^2\frac{\partial U}{\partial Y}.$$

For the velocity profile obeying the „1/7“law, $dU/dY = d/dY(Y^{1/7}) = Y^{-6/7}/7$. Hence, taking into account that $Y = 1 - R$, obtain $Z = 1 + 0.014\mathrm{Re}\sqrt{\mathrm{Pr}}(1-R)^{8/7}$.

Thus, the Lyonś integral is

$$\frac{1}{\mathrm{Nu}} = 0.54\int_0^{R1} \frac{R^3}{1 + 0.014\mathrm{Re}\sqrt{\mathrm{Pr}}(1-R)^{8/7}} dR$$

$$+ 0.54\int_{R1}^1 R^3 dR. \quad (1)$$

The available publications typically use the boundary layer separation according to dynamic parameters. However, in considering the process of heat transfer it seems reasonable to use their thermal analogues. In this connection, in this work we use the „thermal sublayer“instead of „viscous sublayer“İn the framework of the two-layer model, the viscous sublayer thickness is assumed to be restricted by $yU_\tau/\nu < 10$. Let us define the viscous sublayer boundary by using Blasius formula $\xi = 0.3164/\mathrm{Re}^{0.25}$. Then the dynamic velocity is $U_\tau = (\tau/\rho)^{0.5} = (c_f/2)^{0.5}U_m = 0.2U_m/\mathrm{Re}^{1/8}$, where the friction coefficient is $c_f = \xi/4$. Hence, the viscous sublayer thickness is $Y < 10\nu\mathrm{Re}^{1/8}/(r_0 \cdot 0.2U_m)$ or $Y < 100/\mathrm{Re}^{7/8}$. Based on assumption (5), assume that the heat transfer in the sublayer with thickness $Y < 100/(\mathrm{Re}^{7/8}\mathrm{Pr}^{1/3})$ proceeds only due to the molecular thermal conductivity, while that in the flow core is due to both the molecular and turbulent conductivities. Thus, the thermal sublayer boundary is $R1 = 1 - 100/(\mathrm{Re}^{7/8}\mathrm{Pr}^{1/3})$ or $Y1 = 100/(\mathrm{Re}^{7/8}\mathrm{Pr}^{1/3})$. This boundary may be represented as $Y1 = 100/\left((\mathrm{Re}\sqrt{\mathrm{Pr}})^{7/8}\mathrm{Pr}^{-0.1}\right)$. This relation implies the sublayer thickness dependence on both $\mathrm{Re}\sqrt{\mathrm{Pr}}$ and, separately, Pr. Here the integrands are, in essence, distributions of thermal resistance (in arbitrary units) along the pipe radius, while the integral itself represents the total thermal resistance of the flow. Therewith, the final Lyonś integral expression turns out to be quite simple (free of cumbersome empirical relations). The results of numerical integration of relation (1) for medium-Prandtl-number fluids ($\mathrm{Pr}^{-0.1} \sim 1$) are presented in the Table and Figs. 1, 2. As shown in Fig. 1, the maximum thermal resistances get achieved in the channel near-wall region at $Y \to 0$ ($R \to 1$). When $\mathrm{Re}\sqrt{\mathrm{Pr}}$ increases, the ratio between thermal resistances of the thermal sublayer and flow core varies considerably: from 77 and 23% at $\mathrm{Re}\sqrt{\mathrm{Pr}} = 5 \cdot 10^3$ to 40 and 60% at $\mathrm{Re}\sqrt{\mathrm{Pr}} = 10^7$.

Results of calculation

$\mathrm{Re}\sqrt{\mathrm{Pr}}$	$Y1$	Nu
$5 \cdot 10^3$	$5.8 \cdot 10^{-2}$	24
10^4	$3.2 \cdot 10^{-2}$	39
10^5	$4.2 \cdot 10^{-3}$	218
10^6	$5.6 \cdot 10^{-4}$	1325
10^7	$7.5 \cdot 10^{-5}$	8894

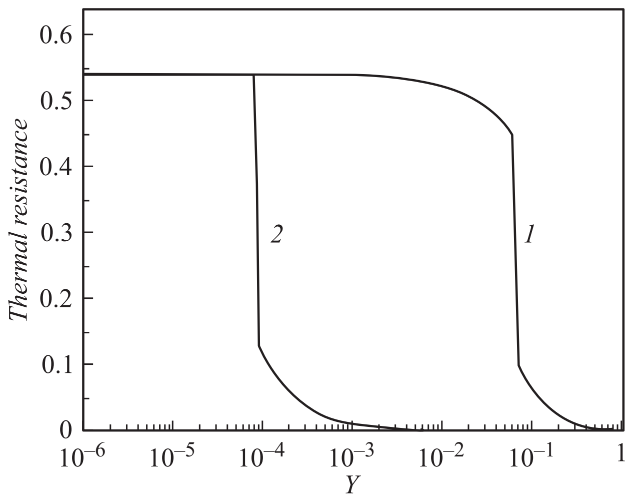

Figure 1. Thermal resistance at $\mathrm{Re}\sqrt{\mathrm{Pr}} = 5 \cdot 10^3$ (*1*) and 10^7 (*2*).

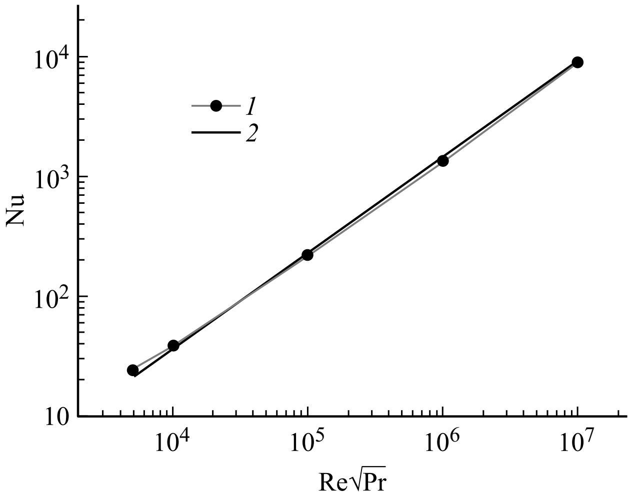

Figure 2. Heat-transfer coefficient. *1* — integral (1), *2* — $\mathrm{Nu} = 0.023\mathrm{Re}^{0.8}\mathrm{Pr}^{0.4}$.

The obtained data on the heat-transfer coefficient (Fig. 2) exhibits a clear association with the empirical Dittus–Boelter relation $\left(\mathrm{Nu} = 0.023(\mathrm{Re}\sqrt{\mathrm{Pr}})^{0.8}\right)$. Note that, formally, in both relations the heat transfer is determined by the same parameter $\mathrm{Re}\sqrt{\mathrm{Pr}}$. The existing deviations of about 8% are, probably, associated with the factors not taken into account here. In general, we may consider the achieved agreement to be quite good and made assumptions to be physically justified at least for $\mathrm{Pr}^{-0.1} \sim 1$. In this case, the Lyonś integral defines the heat-transfer coefficient as a function of turbulent thermal conductivity and thermal sublayer thickness (thermal resistance): $\mathrm{Nu} = f(\lambda_t, y1)$, where, in turn, $\lambda_t = f_1(\mathrm{Re}, \mathrm{Pr})$ and $y1 = f_2(\mathrm{Re}, \mathrm{Pr})$.

Probably, the proposed approach will be also useful in assessing heat transfer in more complex flows, e. g. in channels with different intensifiers.

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Conflict of interests

The author declares that he has no conflict of interests.

References

[1] I.E. Lobanov, Vestn. Angar. gos. tekhn. un-ta, **1** (14), 52 (2020). DOI: 10.36629/2686-777X-2020-1-14-52-59 (in Russian)
[2] D. Taler, Procedia Eng., **157**, 148 (2016). DOI: 10.1016/j.proeng.2016.08.350
[3] P.L. Kirillov, M.I. Terentéva, *Turbulentnoe chislo Prandlya (istoriya i sovremennost)́* (FEI, Obninsk, 2017). https://www.ippe.ru/images/publications/preprints/2018/3271_kirillov.pdf (in Russian)
[4] R.I. Soziev, High Temp., **33** (2), 250 (1995).

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