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Harmonic Analysis of the Librational Model of the Lunar Liquid Core

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A method for constructing a theory of rotation of the Moon, which has a liquid core, is presented. The solution to the problem is carried out within the framework of the Poincare method, which allows one to consider the rotational motion of a body with a cavity filled with a homogeneous incompressible liquid located in the gravitational field. A mathematical apparatus for solving a similar problem for the simplified model of a two-layer Moon is given.

Keywords: lunar core, physical libration of the Moon, Poincare method.

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The task of correction for the Moon's core effect on its rotation became relevant after some discoveries had been made in 70–80 years of the last century using magnetometric, seismological, and laser measurements with the help of the ground-based and Moon-based equipment [1]. Studies have been conducted that incontrovertibly evidenced of the complex stratigraphy of the Moon body and possible existence of a fluid core inside the Moon. At the same time, the absence of a strong magnetic field near the Moon indicated that if there was a core, it was small in size and/or rotating slowly [2]. A great contribution to determining the parameters of a possible Moon's core was made by NASA scientists supervised by James Williams [3], when the lunar core of a certain chemical composition was incorporated in the physical libration theory. By computer simulation of libration theory parameters and verification of the results obtained using sufficiently long series of laser observations, it was concluded that the size of the core was estimated within 300–600 km, depending on the chemical composition: a pure iron core or the core with an eutectic composition Fe–FeS. This data was important, yet, it was only an indirect evidence of the Moon's core existence. However, in 2011 by applying the new methods of Apollo seismograms (Weber et al. [4]) it was concluded that the Moon, similar to the Earth, has a hot metallic core. Its diameter makes approximately 330–360 km, and it is surrounded by a partially molten shell about 480 km in diameter, with a solid iron core about 240 km in diameter inside. Thus, for the first time, direct evidence was obtained about the Moon's core and its two-layered structure. According to estimates [5] the weight of the Moon's core amounts to 1.63–2.06% of the total Moon weight, and its radius — about 20% of the Moon radius. According to the re-processed seismologic data [5], the core is featuring a thin solid shell and a fluid component with a moment of inertia about 70% of the total Moon's core moment of inertia [6]. In this paper, a system of equations is obtained for the particle motion velocity rotor in the reference coordinate system associated with the fluid itself, and the additional velocity due to the rotation of the entire celestial body. This system of equations was solved by numerical computation. The initial values were calculated based on a theory of physical libration for the solid Moon. A spectrum of residual differences was obtained for the three libration components and an attempt was made to analyze the frequency spectrum in order to identify the frequencies at which the fluid core is observed.

It's quite difficult to build a physical libration theory for the three-layer core model, let alone such structure demonstrates very weak observable manifestations. Therefore, the theoretical description of the Moon's rotation with the core was implemented in various theories on simplified core models, and then the contribution of the core to libration was estimated using computer modeling methods by comparing it either directly with laser observations or with high-precision theories of the Moon physical libration, such as DE or ELP. When constructing our theory of physical libration, we carried out a comparison with Rumbaugh and Williams semi-empirical series [7].

In this context, a significant contribution into the study of the two-layered Moon rotation was made by Barkin et all. [8]. Here, for the first time, an analytical libration theory was designed for a two-layered Moon structure: in this theory a core model was used with such parameters as — size, weight, moments of inertia — estimated based on processing of seismological and laser data in papers [4,7,9], as well as based on the gravimetric data from Selene mission. According to the analytical data from theory [8], the hydrodynamic influence of the fluid core on the Moon's physical libration was such that the resulting solution revealed new harmonics, the frequencies of which coincided with frequencies of the semi-empirical series [7], the physical nature of which couldn't be explained by Rumbaugh and Williams. This is a crucial point: the fact that the frequencies obtained from the theory coincide with the observational data opens up new possibilities for determining the parameters of the core.

In our study, we partially follow the ideas from [8], where Poincare method was used in the analysis, but we have our own approach to describe the Moon's rotation parameters and, accordingly, a different way to make the libration equations. Poincare method makes it possible to consider in interaction the rotational motion of a body with a cavity filled with a homogeneous incompressible fluid that is in the gravitational field [10].

To build a theory of rotation of a two-layered celestial body, we consider its simplified model: the dynamic shapes of both the outer solid shell, and inner fluid cavities have the shape of a circular ellipsoid, i.e., due to rotation, compression occurs only along the polar axis, and a circle is located in the equator of such a body. For this model the equator moments of inertia $A = B$ and $A_1 = B_1$ will be equal. We took the core parameters from paper [8] to compare our solution with the results of analytical theory. The second simplification is the assumption that at the initial moment of time, the moments of inertia of the solid Moon and the fluid cavity are co-directional (Fig. 1). This assumption has a right to exist, since the viscosity of the outer Moon's core is still quite high — $(2.07 \pm 1.03) \cdot 10^{17}$ Pa · s [10], which indicates that the core material is not very fluid, the core will be less mobile inside the solid shell and the discrepancy of trihedra of the solid shell core's axes of inertia will be insufficient. In other words, we assume that the angular velocity of the ellipsoidal core coincides with the angular velocity of rotation of the entire body. This assumption allows us to calculate the initial values of the components of the core

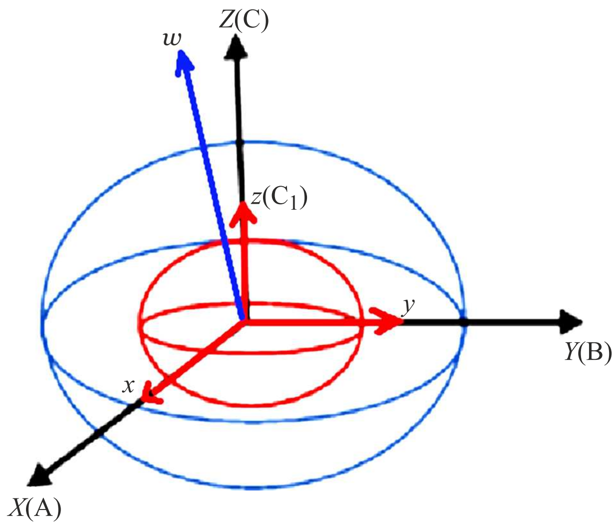

Figure 1. A two-layer model of a celestial body, with circular planes of the equators. The moving coordinate system (x, yz) coincides with the major moments of inertia of the fluid cavity. Stationary (X, Y, Z) — DCS of the Moon.

rotation velocity based on the solid Moon theory that we have already designed [6].

Henri Poincare was a man who largely contributed to the development of mathematical physics and mechanics, especially to the study of fluid dynamics and chaos theory. One of the aspects of his work was the study of the velocity vector field of fluid components in a moving coordinate system. It is defined as a description of the velocity of fluid particles at each moment of time at a given point in space in a moving coordinate system. For a rotating fluid cavity, a moving coordinate system (x, y, z) is usually assumed to be a system that can rotate with the object. In our case it is the tetrahedron of the inertia major axes corresponding to the moment of inertia (A_1, B_1, C_1). This system is called a dynamic coordinate system (DCS) of the core.

The equations describing the vector field of fluid rotation velocities inside an ellipsoidal cavity are expressed as

$$\frac{\vartheta_x}{a} = q_y \frac{z}{c} - q_z \frac{y}{a},$$

$$\frac{\vartheta_y}{a} = q_z \frac{x}{a} - q_x \frac{z}{c},$$

$$\frac{\vartheta_z}{c} = q_x \frac{y}{a} - q_y \frac{x}{a}.$$

Here, $\vartheta_x, \vartheta_y, \vartheta_z$ — components of the fluid velocities in a moving coordinate system; $a, b = a, c$ — dimensions of (half-axis) of ellipsoid plane; q_x, q_y, q_z — time functions representing the angular velocity components $\mathbf{q}$ of the fluid rotation.

The origin of the moving coordinate system (x, y, z) is associated with the Moon center of gravity. Its motion is studied in a stationary coordinate system, as which it is convenient for us to take the system of the main moments of inertia of the Moon's mantle — A, B, C — mantle DCS.

Let's denote the rotor of the particle's translational velocity vector $\mathbf{V}$ in a stationary coordinate system of a fluid as $\xi = \mathrm{rot}(\mathbf{V})$, then, if we go to a rotating coordinate system well get that this vector ξ may be represented as the following sum $\xi = \xi^{(1)} + 2\mathbf{w}$, where $\mathbf{w}$ — angular rotation velocity of a celestial body, and $\xi^{(1)}$ — some vector function of dimensions $a, b = a, c$ and velocity $\mathbf{q}$. In our model, we assume that the angular velocity of the ellipsoidal core coincides with the angular velocity $\mathbf{w}$ of rotation of the entire body.

The physical meaning of component $\xi^{(1)}$ is that it represents the rotor of the particle velocity vector in a coordinate system associated with the fluid itself. This is a kind of local contribution to the overall rotational motion associated with the characteristics of the fluid itself (for example, with the structure and movement inside the fluid). Since the angular velocity $\mathbf{w}$ doesn't depend on the internal dynamics of the fluid itself, it is added as a clear„rotational twist" to the overall rotor. The term „rotational twist" here implies an effect which occurs when the angular velocity of the fluid is added to the rotor irrespective of the dynamics of the fluid itself. This means that the angular velocity is itself

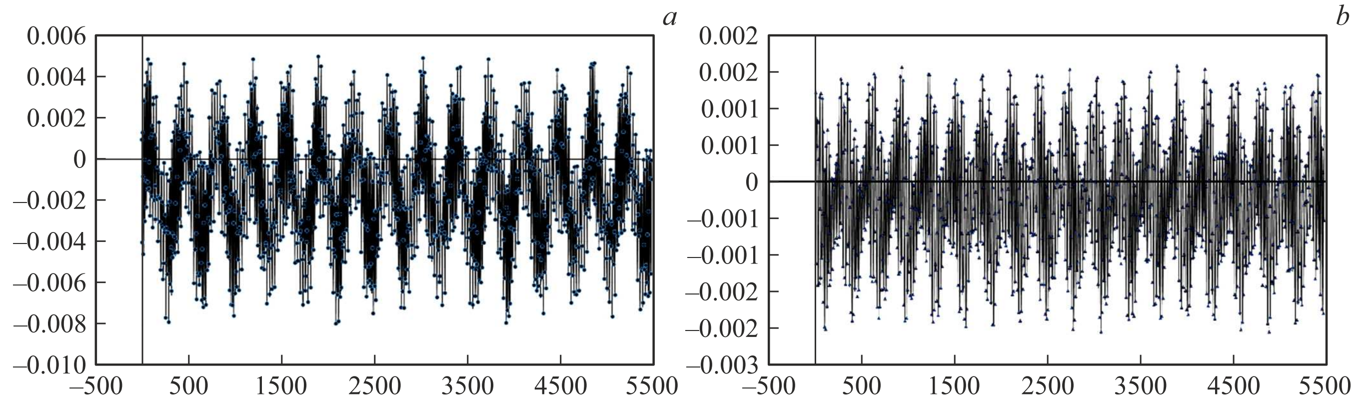

Figure 2. Diagrams of residual differences in libration angles over longitude μ (a) and over latitude ν (b) for the two-layered and solid-state models compared.

an „added" motion that overlaps with the existing rotation. Thus, $\boldsymbol{\xi}^{(1)}$ takes into account all internal motions inside the fluid that are not directly related to the rotation of the entire body as a whole.

Therefore, when switching to a rotating coordinate system, the velocity rotor $\mathbf{V}$ is divided into two components:

1) $\boldsymbol{\xi}^{(1)}$ — local contribution depending on the characteristic of the fluid itself and on its motion;

2) $2\mathbf{w}$ — contribution related to the angular velocity of the entire system.

Let's write down Newton's equations for the fluid motion

$$\rho\left(\frac{d\mathbf{V}}{dt} + (\mathbf{V}, \boldsymbol{\nabla})\mathbf{V}\right) = -\boldsymbol{\nabla} p$$

in the stationary coordinate system, where p — fluid pressure from the external force acting on the fluid. Having first taken the rotor from the entire equation, we obtain the following components of the derivative of vector $\boldsymbol{\xi}^{(1)}$, describing the nature of the fluid elements motion:

$$\dot\xi_x^{(1)} + 2\dot w_x - \alpha\xi_y^{(1)} w_z + \frac{1-\alpha}{2}\,\xi_y^{(1)}\xi_z^{(1)} + w_y\xi_z^{(1)} = 0,$$

$$\dot\xi_y^{(1)} + 2\dot w_y + \alpha\xi_x^{(1)} w_z - \frac{1-\alpha}{2}\,\xi_x^{(1)}\xi_z^{(1)} - w_x\xi_z^{(1)} = 0,$$

$$\dot\xi_z^{(1)} + 2\dot w_z + (2-\alpha)\big[\xi_y^{(1)} w_x - \xi_x^{(1)} w_y\big] = 0, \qquad (1)$$

where the following notation is introduced:

$$\alpha = \frac{2a^2}{a^2+c^2}, \quad q_x = \frac{\xi_x^{(1)}}{c/a + a/c} = \frac{c}{a}\,\frac{\alpha}{2}\,\xi_x^{(1)},$$

$$q_y = \frac{c}{a}\,\frac{\alpha}{2}\,\xi_y^{(1)}, \quad q_z = \frac{\xi_z^{(1)}}{2}. \qquad (2)$$

For a complete solution, we need to add the rotation equations. For this purpose, we use the Euler–Liouville $\dot{\mathbf{M}} + [\mathbf{wM}] = \mathbf{L}$ equations for a moving coordinate system, where the angular momentum (in stationary CS), taking into account the fluid, is expressed as follows:

$$M_i = \sum_k I_{ki} w_k + \iiint \rho[\mathbf{r}\boldsymbol{\vartheta}]dV. \qquad (3)$$

After switching to a rotating coordinate system and simple transformations, we obtain

$$A\dot w_x + \frac{1}{2}\,\dot\xi_x^{(1)}\tilde A + (C-A)w_z w_y + \frac{1}{2}\,\xi_z^{(1)} w_y C_1 - \frac{1}{2}\,\xi_y^{(1)} w_z \tilde A = L_x,$$

$$A\dot w_y + \frac{1}{2}\,\dot\xi_y^{(1)}\tilde A - (C-A)w_z w_x - \frac{1}{2}\,\xi_z^{(1)} w_x C_1 + \frac{1}{2}\,\xi_x^{(1)} w_z \tilde A = L_y,$$

$$C\dot w_z + \frac{1}{2}\,\dot\xi_z^{(1)} C_1 + \frac{1}{2}\tilde A\big(\xi_y^{(1)} w_x - \xi_x^{(1)} w_y\big) = L_z, \qquad (4)$$

where $\tilde A = C_1(1-\alpha) + \alpha A_1$. The moment of forces $\mathbf{L}$ determines the behavior of the system related to the gravitational interaction between the studied body and external disturbing bodies, primarily with the Earth and the Sun. The projections of the moments of forces L_x, L_y, L_z — are functions of time, and they depend on the potential of the Moon's gravitational field. The expression of this potential is not given due to its bulkiness, it is presented in paper [6]. The resulting systems (1) and (4) describe the rotation of an ellipsoidal body filled with an incompressible fluid.

Joint system solution (1) and (4) numerically allowed us to find a solution for the components of the angular velocity of the Moon's rotation $\mathbf{w}$, which in this solution already carries the influence of a liquid cavity rotating inside a solid shell. We calculated the initial values for the numerical integration of these systems according to our theory [6], which is quite reasonable under our accepted constraints on the core model.

In libration theory, position of a rotating body in the inertial coordinate system is described by the libration angles [6]. To assess how the change in angular velocity affected the vibrational angles and thereby to estimate the contribution of the liquid core to the rotation of the Moon, we use the system of Euler kinematic equations obtained in [6]:

$$w_x = -\dot M \sin\nu - \dot\pi,$$

$$w_y = -\dot M \cos\nu \sin\pi + \dot\nu \cos\pi,$$

$$w_z = \dot M \cos\nu \cos\pi + \dot\nu \sin\pi. \qquad (5)$$

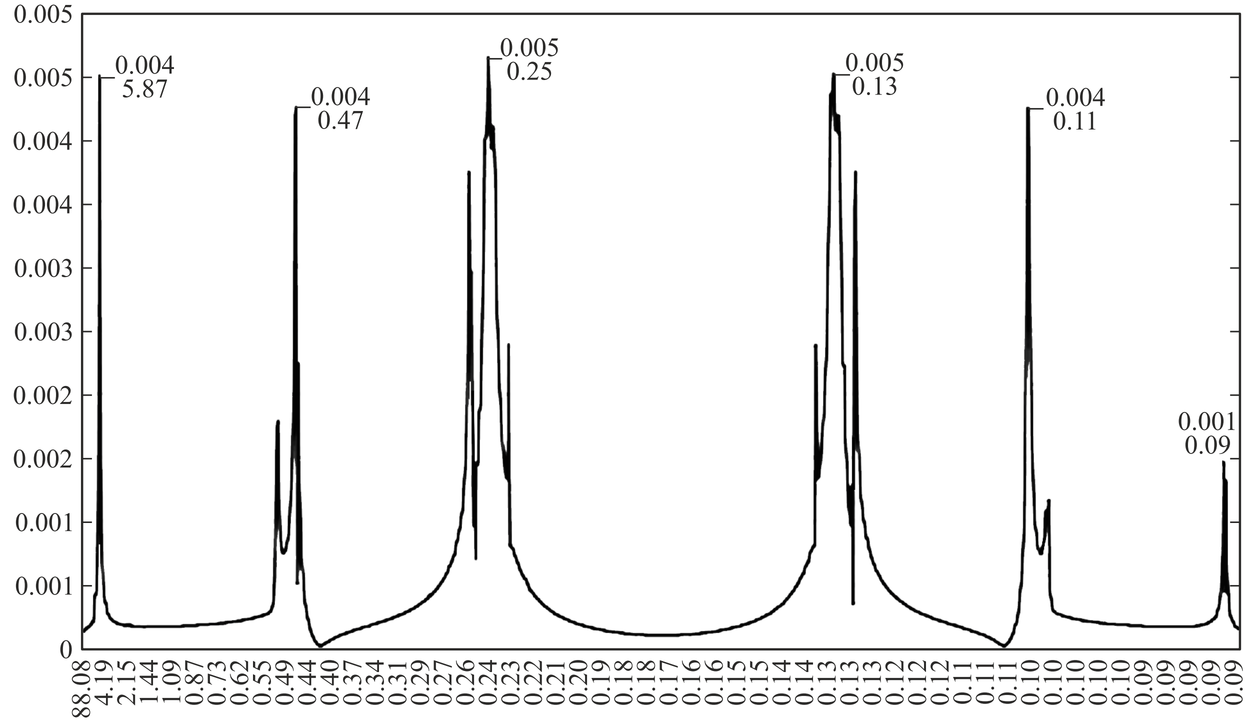

Figure 3. Spectrum of residual differences in the libration angle μ. The Y-axis is set in arc seconds, and the X-axis is set in years.

In system (5) the variables ($\mu = M - \overline{l}_{☾}$, ν, π) — are the libration angles, delineating the Moon DCS position relative to the ecliptic and average direction towards the Earth $\overline{l}_{☾}$. The system of differential equations (5) was solved by the numerical Runge–Kutta method of the 4th order. The initial values for angles and their derivatives were also taken from the theory [6].

After solving the systems of equations (1), (4) and (5) we obtained the numerical values of libration angles (μ^c, ν^c, π^c), that already include the effect of a liquid core. Fig. 2 illustrates the residual differences when comparing (μ^c, ν^c) with the angles obtained for a solid-state Moon [6] for a period of 15 years. For the angle π^c the pattern is similar to the angle ν^c, we therefore didn't attach the third curve to it.

The first thing to pay attention to — the magnitude of the residual differences for all angles is small and doesn't exceed 5 ms, which is almost a limited accuracy for the integration method used. Second, we performed a frequency analysis for the obtained residual difference spectra in order to identify the frequencies at which the influence of the liquid core is most pronounced. We expected them to coincide with the frequencies obtained in paper [8] or with the frequencies of Un members in series [7]. Fig. 3 gives the results of the frequency analysis for the libration over longitude. Unfortunately, the spectrum of the identified frequencies is consistent with neither the data from [7], nor the data from [8]. The selected periods (5.87 years, 0.47 years, and etc.) are not observed in the librational series of the Moon, and do not seem to have a physical meaning. The situation is the same for the latitude components.

Nevertheless, we believe that the work done has not been done in vain. This is only our first attempt to consider the rotation of a two-layered Moon, and the technique we have described here is quite realistic: we have obtained a solution that, as expected, is not significantly different from the solution of [6]. Our task now is to continue development of the presented approach to account for the influence of the core on the Moon rotation. To do this, we plan to complicate our initial model, bringing it closer to a more realistic picture of rotation of a two-layered Moon, and carefully check the recording of all systems of equations at both mathematical and software levels. The theory obtained will make it possible to refine the core parameters by computer modeling based on the theory being developed.

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Conflict of interest

The authors declare that they have no conflict of interest.

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