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# Stability of potential rotation of ideal fluid

© D.A. Shalybkov

Ioffe Institute, St. Petersburg, Russia
E-mail: shalybkov@yandex.ru

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The stability of potential rotation of ideal incopressible homogenious fluid is considered in linear approximation. It is shown, that for rigid boundaries there are no asymmetric modes as stable so unstable. There are only stable singular modes.

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The problem of stability of an ideal incompressible rotating fluid is a classical problem of hydrodynamics [1–4]. It is common knowledge that solid-body rotation is stable with respect to any perturbations in the linear approximation (see, e.g., [5]).

The stability condition for an ideal incompressible rotating fluid with uniform density with respect to axisymmetric perturbations was obtained by Rayleigh [6]. The Rayleigh condition in a cylindrical coordinate system $(r, \varphi, z)$ takes the form

$$\frac{1}{r^3}\frac{d}{dr}(r^2\Omega) \geqslant 0, \tag{1}$$

where $\Omega$ is the angular rotation velocity. It has been demonstrated later in [7] that condition (1) is necessary and sufficient for stability. The motion of an incompressible ideal fluid with uniform density $\rho$ is characterized by the Euler and continuity equations

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P, \qquad \mathrm{div}\mathbf{U} = 0, \tag{2}$$

where $\mathbf{U}$ is velocity and $P$ is pressure. A cylindrical coordinate system $(r, \varphi, z)$ is convenient for a rotating fluid. Its velocity in this system is written as

$$U = (0, r\Omega(r), 0), \tag{3}$$

where $\Omega(r)$ is the angular rotation velocity. In the case of an ideal fluid, this velocity is an arbitrary (sufficiently smooth) function of radius that satisfies Eq. (2) and the boundary conditions. In the stationary case, Eq. (2) for velocity (3) takes the form

$$\Omega^2 r = \frac{1}{\rho}\nabla P. \tag{4}$$

The method of small perturbations is used to study stability in the linear approximation. The solution is then presented as

$$U + u = \big(u_r(r, \varphi, z, t), r\Omega(r) + u_\varphi(r, \varphi, z, t), u_z(r, \varphi, z, t)\big),$$
$$P + p = P(r) + p(r, \varphi, z, t), \tag{5}$$

where quantities $u_r$, $u_\varphi$, $u_z$, and $p$ are small compared to the unperturbed ones. Inserting expressions (5) into system (2) and retaining only the terms linear in perturbed quantities, one obtains a linear system of equations for perturbed quantities with coefficients that depend on radial coordinate $r$ only. The solution may then be presented as a sum of normal modes of the form

$$F = F(r)\exp\big(i(m\varphi + kz + \omega t)\big), \tag{6}$$

where $F(r)$ is an arbitrary sought-for function. With the geometry of the problem taken into account, axial number $k$ may assume arbitrary real values, azimuthal number $m$ is an arbitrary integer, and increment $\omega$ is an arbitrary complex number. The expansion in normal modes (6) transforms a three-dimensional problem into a one-dimensional one. If natural frequencies $\omega$ have only positive imaginary parts, the flow is linearly stable. If at least one natural frequency with a negative imaginary part is present, the flow is unstable. Simple transformations allow one to reduce the linear system to a single second-order equation for radial velocity $u_r$

$$\omega_d^2\frac{d}{dr^2}\left[\frac{r^2}{k^2r^2+m^2}\frac{1}{r}\frac{d}{dr}(ru_r)\right] - \omega_d^2 u_r$$
$$-\omega_d mr\frac{d}{dr}\left[\frac{r^2}{k^2r^2+m^2}\frac{d}{dr}(r^2\Omega)\right]u_r$$
$$+\frac{k^2r^2}{k^2r^2+m^2}\frac{1}{r^3}\frac{d}{dr}\big(r^2\Omega\big)^2 ur = 0, \tag{7}$$

where $\omega_d = \omega + m\Omega$ is the Doppler increment. Note that Eq. (7) matches Eq. (18) from [5] if the axial velocity is zero ($W = 0$ in the notation of [5]). It is important for us that Eq. (7) is derived without division by $\omega_d$, which may turn to zero at point $r_0$ such that

$$\omega = -m\Omega(r_0). \tag{8}$$

Boundary conditions are needed to complete the formulation of the problem. In the present study, we consider a region that is not limited in axial coordinate $z$, extending from inner radius $r_{in} > 0$ (as will be shown below, potential rotation cannot reach the axis of rotation; see (10)) to outer radius $r_{out} < \infty$ and occupying the entire angular sector of $0 \leqslant \varphi \leqslant 2\pi$. Impermeability boundary conditions are used:

$$u_r(r_{in}) = u_r(r_{out}) = 0. \tag{9}$$

Equation (7) with boundary conditions (9) constitutes an eigenvalue problem for frequency $\omega$. It is clear that condition (8) may be satisfied only if the imaginary part of $\omega$ is zero. In addition, it is assumed in the derivation of (7) that numbers $k$ and $m$ cannot be both equal to zero. It is easy to verify that boundary condition (9) is not satisfied if $k = m = 0$. Let us demonstrate that there are no non-trivial solutions to Eq. (7) for potential rotation of a fluid if condition (8) is not satisfied. Rotation is potential if $\mathrm{rot}\mathbf{U} = 0$. This condition for velocity (3) implies that

$$\Omega = Cr^{-2}, \tag{10}$$

where $C$ is a constant determined by boundary conditions. Note that, according to (10), potential rotation cannot extend to the axis of rotation. In addition, according to (1), potential rotation is stable with respect to axisymmetric perturbations and (e.g., in the case of a cylindrical Couette flow) separates stable flows from unstable ones (see, e.g., [1]).

Let us assume that condition (8) is not satisfied at $r_{in} \leqslant r \leqslant r_{out}$ and demonstrate that Eq. (7) has no non-trivial solutions in this case. Indeed, suppose the opposite: there exist non-trivial solutions of Eq. (7) such that condition (8) is not satisfied. Equation (7) may be divided by $\omega_d^2$ in this case. Multiplying the resulting equation by $ru_r^*$, where $u_r^*$ is the complex conjugate function of $u_r$, and integrating by parts with account for boundary conditions (9), we obtain

$$\int_{r_{in}}^{r_{out}} \frac{r}{k^2r^2+m^2}\left|\frac{d}{dr}(ru_r)\right|^2 dr + \int_{r_{in}}^{r_{out}} \left(1 + \frac{mr}{\omega_d}\frac{d}{dr}\left[\frac{r^2}{k^2r^2+m^2}\frac{d}{dr}(r^2\Omega)\right] - \frac{k^2}{k^2r^2+m^2}\frac{1}{r}\frac{d}{dr}\frac{1}{\omega_\mathrm{d}^2}\left(r^2\Omega\right)^2\right) r|u_r|^2 dr = 0. \tag{11}$$

Accordingly, the third and fourth terms are zeroed out for potential flow (10), and expression (11) takes the form

$$\int_{r_{in}}^{r_{out}} \frac{r}{k^2r^2+m^2}\left|\frac{d}{dr}(ru_r)\right|^2 dr + \int_{r_{in}}^{r_{out}} r|u_r|^2 dr = 0. \tag{12}$$

Both terms in expression (12) are positive for non-trivial solution $u_r$. Therefore, this expression cannot be satisfied. Consequently, the initial assumption of existence of a non-trivial solution to Eq. (7) with boundary conditions (9) in the case of violation of condition (8) is incorrect, which is the required result.

Thus, it was demonstrated that the problem of linear stability of rotation of an ideal fluid of the form (3) with respect to asymmetric perturbations is unsolvable for normal modes if this rotation is potential (i.e., has the form (10)).

Note that Eq. (7) may be resolved for singular modes with condition (8) satisfied at a certain point in the flow.

However, as was noted above, the fulfillment of condition (8) implies that the natural frequency is a real number and, consequently, that the corresponding mode is stable. With the stability of potential flow (according to criterion (1)) to axisymmetric perturbations taken into account, potential rotation of an ideal fluid is, in common with solid-body rotation, stable with respect to any perturbations.

It should be emphasized that the condition of fluid homogeneity is essential. For example, in the case of stable vertical density stratification, potential rotation becomes unstable with respect to asymmetric perturbations (see, e.g., [8–10]).

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### Conflict of interest

The author declares that he has no conflict of interest.

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