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# Justification of the Bodner–Partom empirical constitutive equation under quasi-static deformation of materials within the framework of acoustoplastic effect

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The processes of elastic and plastic deformation of materials are considered within the framework of the modified model of the acoustoplastic effect. The conditions of reducing it to the widely used empirical Bodner–Partom model for describing experimental stress–strain curves are analyzed. The conditions of applicability of this empirical model are noted. The relationship between the parameters used in the Bodner–Partom model and such material parameters as the internal friction stress, the activation volume of defects, their relaxation time and their equilibrium concentration, as well as with the parameter characterizing the degree of interaction of defects is determined.

**Keywords:** mechanical stress, plastic strain rate, acoustoplastic effect, defects.

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## 1. Introduction

A number of stress-strain properties of a material are currently determined using loading machines to set a particular strain rate and to record the applied stress corresponding to this strain [1,2]. At the beginning of the elastic segment of the stress–strain curve, its linear part is defined by Young's modulus of the material. Strain analysis in the plastic strain area determines the yield stress and strain hardening variables. Complexity of physical processes flowing in the material during transition from elastic to plastic strain requires that quite sophisticated physical and mechanical strain models be developed taking in account defect formation, interaction and propagation processes. The lack of such models facilitated a widespread occurrence of a number of empirical models describing the correlation between stress and strain in the plastic region. When used for interpreting experimental results, these models provide quantitative characteristics for describing inelastic behavior of a material using a set of variables, physical meaning of which is often not fully understood.

As shown in [3], an acoustoplastic material strain model is used to substantiate a number of empirical strain models and to explore the physical and mechanical nature of variables used for the models. Efficiency of such approach was demonstrated for the Johnson–Cook empirical model [4–6] and more specific Voce [7,8] and Hollomon [9,10] models. Conditions, in which the acoustoplastic model data allows the above-mentioned empirical models to be used for describing the strain-stress correlation in the plastic region, were established.

In 1975, Bodner and Partom proposed an empirical model [11] based on the approach other than the Johnson–Cook model. It is currently being developed in the same way as the Johnson–Cook model and widely used for describing stress–strain curves [12,13].

## 2. Physical and mechanical justification

Let's provide a physical and mechanical substantiation of the Bodner–Partom model in terms of the acoustoplastic effect, consider potential Bodner–Partom model substantiation within the acoustoplastic effect model and give physical interpretation for the model variables. For this, the main equations will be used for describing the stress behavior on a sample in terms of the acoustoplastic effect in nonstationary strain [14,15]

$$\frac{1}{E}\frac{\partial\sigma}{\partial t} = \dot{\varepsilon} - \dot{\varepsilon}_p, \tag{1}$$

where $E$ is Young's modulus of material, $\dot{\varepsilon}$ is the object's total strain rate from an external source, $\dot{\varepsilon}_p$ is the plastic strain rate of the material.

In order to determine the plastic strain rate $\dot{\varepsilon}_p$, it is usually assumed that defects in a material are generated according to the Arrhenius activation law and may be found from

$$\dot{\varepsilon}_p = \dot{\varepsilon}_v \exp\left(\frac{\Omega\left(\sigma - \sigma_f - \sigma_p(\varepsilon)\right)}{k_B T}\right), \tag{2}$$

where $\sigma_f$ is the stress induced by internal friction for defects; $\sigma_p$ is the sample stress associated with defects generated in the sample; the preexponential factor $\dot{\varepsilon}_v$ describes the material strain rate due to dislocation displacement, and is generally assumed to be constant; $\Omega$ is the activation

volume of a defect; $k_B$ is the Boltzmann constant; $T$ is the sample temperature.

Traditional approach to the acoustoplastic effect usually uses equations (1) and (2). Dependence of the strain hardening $\sigma_p$ on $\varepsilon$ has to be introduced empirically [14]. In [3,16–18], it was shown that the form of this dependence may be determined from the kinetic equation for defect concentration taking into account defect relaxation properties and interaction behavior. Analysis of this approach determines a stress-strain correlation in the plastic region using the following relation

$$\sigma_p(\varepsilon_p) \simeq \sigma_f \Omega n(\varepsilon_p) - n(\varepsilon_p) e_p, \tag{3}$$

where $e_p$ is the plastic strain energy per defect.

Expression (3) is a generalization of the known relation $\sigma = E\Omega n$ [19] that considers the stress in the material's elastic region in the presence of defects with the concentration $n(\varepsilon)$. General stress on the sample in the plastic strain region is determined by the expression $\sigma(\varepsilon_p) \cong \sigma_f + \sigma_p(\varepsilon_p)$. When considering material behavior in the plastic region, $\sigma_f$ is more suitable than $E$. The last term in expression (3) reflects the stress variation in a sample due to the energy release $e_p$ near the defect.

In [3], it was shown that solution to equation (1) with respect to $\sigma$ using expressions (2) and (3) may be given by

$$\sigma(\varepsilon) = E\varepsilon - \frac{k_B T}{\Omega} \ln\left[1 + \frac{\Omega E}{k_B T} \dot{\varepsilon}_v \int_0^{\varepsilon} d\varepsilon' \frac{1}{\dot{\varepsilon}'} \times \exp\left(\frac{\Omega(E\varepsilon' - \sigma_f - \Omega\sigma_f n(\varepsilon') + n(\varepsilon') e_p(\varepsilon'))}{k_B T}\right)\right], \tag{4}$$

where $n(\varepsilon)$, according to [3,16], varies as follows

$$n(\varepsilon) = n_r\left[1 - \exp\left(-(\varepsilon/\dot{\varepsilon}_v \tau)^{\beta}\right)\right],$$

$n_r$ is the equilibrium concentration of defects, $\tau$ is the defect relaxation time, $\beta$ is the coefficient characterizing the degree of defect interaction [20]. Equation (4) may be treated as a sufficiently general equation of state of a material in strain both in the elastic and plastic regions.

To obtain the phenomenological Bodner–Partom dependence, behavior of relation (4) in the plastic strain region shall be addressed with $\varepsilon \geq \varepsilon_e$, where $\varepsilon_e$ is the maximum strain corresponding to the elastic region. In this case, the first term in the exponent makes the main contribution, and the other terms may be assumed as slowly varying functions of strain. In these plastic strain region conditions, expression (4) may be rearranged to

$$\sigma(\varepsilon_p) \simeq -\frac{k_B T}{\Omega} \ln \frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_p} - k_B T n(\varepsilon_p) \ln \frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_p} + \sigma_f + \Omega\sigma_f n(\varepsilon_p). \tag{5}$$

In accordance with expression (5), the following relation is obtained for the plastic strain rate

$$\dot{\varepsilon}_p \simeq \dot{\varepsilon}_v \exp\left(\frac{\Omega}{k_B T} \frac{\sigma(\varepsilon_p) - \sigma_f(1 + n(\varepsilon_p)\Omega)}{1 + n(\varepsilon_p)\Omega}\right). \tag{6}$$

Using expression (3), this relation is written as

$$\dot{\varepsilon}_p = \dot{\varepsilon}_v \exp\left(-\frac{\Omega}{k_B T} \frac{n(\varepsilon_p) e_p}{1 + n(\varepsilon_p)\Omega}\right). \tag{7}$$

Relaxation processes in deformation experiments with materials generally run sufficiently slowly, and $\varepsilon_p \leq \dot{\varepsilon}_v \tau$ can be considered to be fulfilled. Then the defect concentration may be determined from $n(\varepsilon_p) \simeq n_r(\varepsilon_p/\dot{\varepsilon}_v\tau)^{\beta}$. We also assume that $e_p = \kappa n(\varepsilon_p)$, where $\kappa$ is a constant. When the defect volume is $\Omega \approx 10^{-28}\,\mathrm{m}^3$, $n\Omega \ll 1$ is fulfilled to sufficiently high defect concentrations $10^{27}\,\mathrm{m}^{-3}$. If the above-mentioned conditions are considered as fulfilled, then for (7) we obtain

$$\dot{\varepsilon}_p \simeq \dot{\varepsilon}_v \exp\left(-\frac{\kappa\Omega n_r^2}{k_B T}\left(\frac{\varepsilon_p}{\dot{\varepsilon}_v \tau}\right)^{2\beta}\right). \tag{8}$$

In the right-hand side of equation (8), $\varepsilon_p$ may be expressed in terms of stresses using equation (3). Then, taking into account that $\varepsilon_p \cong \Omega n(\varepsilon_p)$, we obtain

$$\varepsilon_p = \frac{\Omega^2}{2\kappa}\left[\sigma_f - \sqrt{\sigma_f^2 - 4\frac{\kappa}{\Omega^2}(\sigma_p - \sigma_f)}\right]. \tag{9}$$

When $(\sigma_p - \sigma_f)\kappa/(\Omega\sigma_f)^2 \ll 1$ is fulfilled, equation (8) is reduced to

$$\dot{\varepsilon}_p \simeq \dot{\varepsilon}_v \exp\left(-\frac{\kappa\Omega n_r^2}{k_B T}\left(\frac{1}{\dot{\varepsilon}_v \tau}\right)^{2\beta}\left(\frac{\sigma_p - \sigma_f}{\sigma_f}\right)^{2\beta}\right). \tag{10}$$

For a one-dimension case, equation (10) may be given a form used in the Bodner–Portom model. For this, the following function shall be introduced

$$Z = \frac{1}{\dot{\varepsilon}_v \tau}\left(\frac{2\kappa\Omega n_r^2}{k_B T}\right)^{1/2\beta}(\sigma_p - \sigma_f)$$

and $\sigma_f$ shall be assumed as approximately coinciding with the uniaxial stress $\sigma_{11}$ applied to the sample, and $\varepsilon_p$ coincides with component $\varepsilon_{11}^p$ of the plastic strain tensor. Then expression (10) takes a form accepted in the Bodner–Partom model:

$$\dot{\varepsilon}_{11}^p = \frac{2}{\sqrt{3}} D_0 \exp\left(-\frac{1}{2}\left(\frac{Z}{\sigma_{11}}\right)^{2\beta}\right), \tag{11}$$

where $D_0$ is introduced in accordance with notations in [11,12] as accepted in the Bodner–Partom model $D_0 = \frac{\sqrt{3}}{2}\dot{\varepsilon}_v$.

Important comments shall be made regarding the obtained result. One of them refers to the form of dependence of $\dot{\varepsilon}_{11}^p$ on $Z$. Thus, if the Bodner–Partom model establishes the form of this dependence empirically, the acoustoelastic approach substantiates this form quite naturally. Moreover, in the Bodner–Partom model, when function $Z$ is introduced into the equation for $\dot{\varepsilon}_{11}^p$, the nature of this function is still not fully understood. Note that $Z$ at a certain time shall reflect some sample loading history during previous

periods. For this, it is proposed to use the following equation to determine this function without considering thermal reduction processes [11,12]

$$\dot{Z} = -m_1 \dot{W}_p Z + m_1 \dot{W}_p Z_1, \tag{12}$$

where $m_1$ describes the material strain hardening rate; $\dot{W}_p$ is the deformation machine running speed required to achieve the plastic strain of the material; $Z_1$ is the value of $Z$ corresponding to its saturation.

We show that the equation for $Z$ given by (12) may be derived from the kinetic equation for defect concentration used in [3,16]. We assume that the defect rate in the material in plastic strain is set by the deformation machine. Then the kinetic equation for the concentration of defects that are formed during material straining may be written as follows in accordance with [3,16]:

$$\frac{dn}{dt} = -\frac{n}{\tau} + \dot{g}, \tag{13}$$

where $\dot{g}$ is a defect generation rate set by the loading machine.

When we restrict ourselves in the expression for $Z$ to terms linear in defect concentration, then the function is written as

$$Z = \frac{\sigma_f \Omega}{\dot{\varepsilon}_\nu \tau} \left( \frac{2\kappa \Omega n_r^2}{k_B T} \right)^{1/2\beta} n.$$

Using this equality, equation (13) may be rearranged to

$$\frac{dZ}{dt} = -\frac{Z}{\tau} + \frac{\dot{g}'}{\tau'}, \tag{14}$$

where

$$\dot{g}' = \frac{\sigma_f \Omega}{\dot{\varepsilon}_\nu} \left( \frac{2\kappa \Omega n_r^2}{k_B T} \right)^{1/2\beta} \dot{g}.$$

Moreover, when assuming that the defect generation rate is defined by the rate of plastic strain energy arrival to the material in accordance with $1/\tau = m_1 \dot{W}_p$, then equation (14) takes the form of the Bodner–Partom equation for $Z$, whereas $Z_1 = \dot{g}'$. If the defect generation rate $\dot{g}$ is assumed as a constant or slowly varying function over time, then $Z_1$ may be considered as a parameter.

In accordance with the findings, $Z$ is an integral strain parameter. This function also plays the same role in the Bodner–Partom model, however, this model cannot specify the form of this function. The acoustoplasticity model makes this possible and shows that, pursuant to equations (13) and (14), it is closely related to the concentration of defects accumulated in the material during deformation.

## 3. Conclusion

The findings show that the acoustoplastic effect theory together with the kinetic equation used for describing the material defect subsystem behavior makes it possible to substantiate the phenomenological Bodner–Partom model that is widely used in physics and mechanics for describing the stress-strain correlation in the material plasticity region. Previously in [3], it was shown that the acoustoplasticity may be used to substantiate another important phenomenological model — Johnson–Cook model. Consideration of two different phenomenological models within a single acoustoplastic approach makes it possible to identify also some aspects of the use of these models. Thus, while the Johnson–Cook model correlates the plastic strain energy per defect with the plastic strain rate, the Bodner–Partom model determines the plastic strain energy per defect by the defect concentration. In the both cases, the acoustoplastic theory used to review the stress-strain dependence correlates the values of phenomenological variables used in the empirical approaches with material properties such as yield stress, activation volume of defects, equilibrium defect concentration, relaxation time and degree of defect interaction.

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### Conflict of interest

The authors declare that they have no conflict of interest.

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