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Boundary conditions of sliding during the motion of bodies in a dilute emulsion of gas bubbles

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It is shown that the reason for the slip boundary conditions of during the movement of bodies in a viscous and incompressible emulsion of gas bubbles is the hydrodynamic interaction of the bubbles with the surface of the body. The influence of hydrodynamic interaction on the slip parameter ξ increases with an increase in the volume concentration of bubbles φ and with a decrease in the size of the body R compared to the size of the bubbles a. Limit analytical dependences of the slip parameter for a ball moving in an emulsion of identical gas bubbles were obtained: $\xi_{down} < \xi(a/R, \varphi) < \xi_{up}$.

Keywords: slip boundary conditions, Stokes Equations, emulsion of gas bubbles, hydrodynamic interaction.

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The rapid development of energy-saving micro- and nanotechnologies in recent decades has posed a number of new scientific and technical challenges for fundamental science related to mathematical modeling of the properties and dynamics of various microdisperse media [1–3]. Specifically, with an increased concentration of the disperse phase in such media, a complex problem of factoring in the long-range influence of disperse particles on each other (the problem of hydrodynamic interaction [4,5]), which is a variation of the well-known many-body problem from classical mechanics. A similar problem arises when one sets boundary conditions in the mechanics of disperse media.

It is known that heterogeneous media may be analyzed in certain cases as homogeneous [6] by averaging their physical characteristics over a certain representative volume Ω with its dimensions satisfying the $l \ll \Omega^{1/3} \ll L$ condition, where l is the characteristic size of microstructural inhomogeneity of the heterogeneous medium and L is the characteristic size of the problem. On the macroscopic scale of the problem, volume Ω is a physical point of a continuous homogeneous medium, which may be characterized mathematically using the classical differential equations of hydrodynamics. The issue of boundary conditions arises when one integrates the (second-order) Navier–Stokes equation. The first boundary condition is trivial: in the laboratory frame of reference at infinity, the velocity of the medium is zero. The formulation of the second condition at the boundary between a moving body and the medium (or between the medium and a hard wall bounding it) still remains a subject of scientific research and discussions [7–9].

Considerable progress in such research for liquids and gases has been achieved. Specifically, it was demonstrated in numerous experiments that the so-called no-slip condition, wherein liquid velocity $\mathbf{u}_w$ on the wall (here and elsewhere, subscript w denotes wall-related parameters) is equal to local velocity of the wall $\mathbf{u}_w = \mathbf{v}$, where $\mathbf{v}(x_i)$ is the local velocity of the body (wall), is established with high accuracy at the solid–liquid boundary on a macroscopic scale. In contrast, the slip effect was observed in rarefied gases and in microscale liquid flows [1,10]. Navier (1823) has suggested that frictional stress $\boldsymbol{\tau}_s$, which arises due to liquid slip, is, in a first approximation, proportional to the slip velocity: $\boldsymbol{\tau}_s = \beta \mathbf{u}_w$, where β is the slip friction coefficient. In equilibrium, $\boldsymbol{\tau}_s = \boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is the shear stress in liquid adjacent to the wall. Thus, in a Newtonian liquid, wherein the viscosity does not depend on the shear strain rate,

$$\boldsymbol{\tau} = \eta\dot{\gamma} = \eta\left(\frac{\partial u_\tau}{\partial n}\right)_w,$$

where u_τ is the projection of the liquid velocity onto planes perpendicular to the normal, the wall boundary condition takes the form

$$\beta \mathbf{u}_w = \eta\left(\frac{\partial u_\tau}{\partial n}\right)_w, \qquad (1)$$

where $\mathbf{n}$ is the vector of the outward normal to the body surface and η is the dynamic liquid viscosity. It can be seen from (1) that parameter $\lambda = \eta/\beta$ has the dimension of length; it is called the slip length. It is impossible to calculate λ within the macroscopic gas model, since this parameter is determined by the microscopic properties of gas and the surface and their interaction. Maxwell has later (1879) used the methods of the kinetic theory of gases and the mirror-diffusion model of interaction of gas molecules with a smooth surface of a solid to obtain an analytical expression for slip velocity $\mathbf{u}_w$, which takes the following form in the isothermal case of rarefied gas [11]:

$$\mathbf{u}_w = \alpha l_m \left(\frac{\partial u_\tau}{\partial n} \right)_w, \quad (2)$$

where l_m is the mean free path of gas molecules and $\alpha = (2 - \sigma)/\sigma$; σ is the accommodation coefficient of the tangential momentum of gas molecules when they are reflected from a solid surface ($0 < \sigma < 1$). At $\sigma \to 0$, mirror reflection is observed; at $\sigma = 1$, the reflection is diffuse, and the momentum of a molecule is transferred completely to the wall. Equations (1) and (2) make it clear that the αl_m quantity in rarefied gas is analogous to slip length λ in liquid. In the dimensionless case (e.g., flow around a sphere with radius R), this analogy takes the form

$$\alpha \mathrm{Kn} \leftrightarrow \xi, \quad (3)$$

where $\mathrm{Kn} = l_m/R$ is the Knudsen number and $\xi = \lambda/R$ is the dimensionless slip length (parameter).

Thus, Maxwell has proven theoretically the existence of the slip effect in rarefied gases and explained its mechanism [11].

Generally speaking, the no-slip boundary condition in gases is a consequence of hypothesized continuity of a homogeneous medium (when it may be regarded as a continuum with a fair degree of accuracy). If an important scale of the problem (e.g., R) approaches the size of the structural inhomogeneity of the medium (l_m), it was proven by Maxwell [11] that this structural parameter of the medium enters into the solution of the problem through slip boundary conditions (2). Relation (3) makes it evident that when a sphere moves in gas, no-slip boundary conditions are established if Knudsen number $\mathrm{Kn} \ll 1$. A more accurate estimate of applicability of the no-slip condition was given in [12]: $\mathrm{Kn} \leqslant 0.001$.

The pattern in disperse media is similar: when scale L of the problem (the size of a body or the transverse size of a channel with hard walls) is comparable with the structural parameters of the medium (l, a, etc., where a is the radius of a disperse particle), the structural parameters of the disperse medium are expected to enter into the solution of the problem through slip boundary conditions. Indeed, when the above conditions are met, the slip boundary condition is observed reliably in experiments even in disperse media [8,9]. The problem is to understand the physical mechanism of slip in disperse media.

A simplified phenomenological model was presented in [9]. It attributes the effect of slip on an impermeable solid surface to a reduction in effective viscosity of the medium in a thin layer ($h = 2a$) near this surface due to a change in volume concentration of particles in this layer, which decreases from a given value φ to zero. Since the no-slip boundary condition is established on the surface, the authors call it „apparent slip.“ The authors of other papers [13,14] consider „true slip“ and use Navier boundary conditions (1) to investigate the degree of influence of slip parameter ξ on the dynamics of a disperse medium (even with the hydrodynamic interaction of particles factored in). Specifically, the diffusion coefficient of particles in a monodisperse suspension of spherical particles was calculated in [13] to the first order in concentration with account for their hydrodynamic interaction as $D = D_0[1 - \lambda(\xi)\varphi]$. It was demonstrated that, e.g., coefficient $\lambda = 1.56$ at $\xi = 0$, and $\xi = \infty$ corresponds to coefficient $\lambda = 3.50$. It was shown in [14] that the sedimentation rate of a test particle with radius R under the action of gravity in a monodisperse suspension of spherical particles with radius a depends not only on parameter ξ, but also on parameter $\sigma = a/R$.

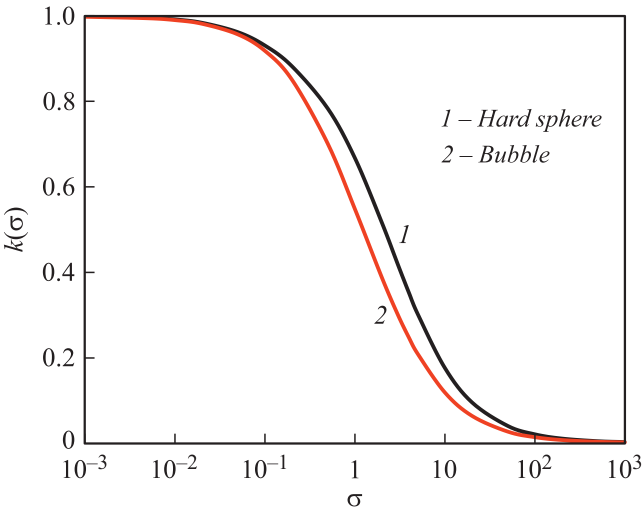

Figure 1. Dependences of coefficients k_s (*hard sphere*, $\xi = 0$) and k_b (*bubble*, $\xi = \infty$) in formula (5) on parameter $\sigma = a/R$.

Thus, slip parameter ξ plays an important role in the dynamics of disperse media. However, as far as we know, no reliable mathematical models predicting the value of relative slip length ξ as a function of the parameters of the internal structure of a disperse medium in a specific problem have been developed to date. This paper provides a justification for the author's hypothesis, which states that the emergence of the slip boundary condition in disperse media may be fully explained by the hydrodynamic interaction of disperse particles with a streamlined hard and smooth surface. This hypothesis allows one to determine the value of parameter ξ in a specific problem.

Let us consider the steady motion of a hard spherical body in an unlimited macroscopically homogeneous and isotropic medium consisting of a homogeneous liquid and identical spherical gas bubbles dispersed chaotically in it. The size of the bubbles and their volume concentration φ do not change; i.e., the processes of bubble coalescence and breakup in the medium are suppressed completely. The disperse medium (an emulsion of gas bubbles) is incompressible and (at $\varphi << 1$) is a Newtonian liquid with dynamic viscosity $\eta = \eta_1(1 + \varphi)$ [15]. The goal is to estimate, in the Stokes approximation, the value of slip parameter ξ that defines Navier boundary condition (1) on the surface of a moving body.

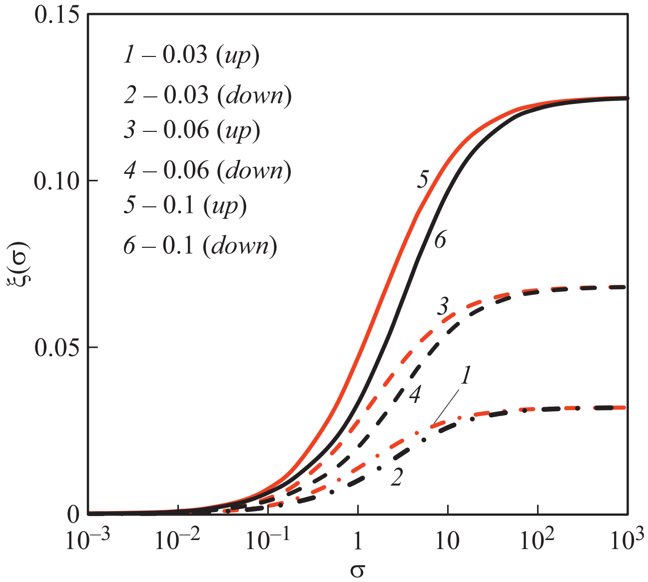

Figure 2. Limits of variation of dimensionless slip length $\xi = \lambda/R$ with parameter $\sigma = a/R$ at volume concentrations of bubbles $\varphi = 0.03$, 0.06, and 0.1.

The problem of motion of a sphere in a homogeneous medium averaged over volume Ω with Navier boundary conditions (1) has an exact solution (Basset, 1888):

$$\mathbf{F} = -\frac{6\pi\eta_1(1+\varphi)R\mathbf{V}}{C}, \tag{4}$$

where $\mathbf{F}$ is the drag force, $\mathbf{V}$ is the body velocity, and $C = (1+3\xi)/(1+2\xi)$. At $\xi = 0$, formula (4) transforms into the exact Stokes solution (1850), which corresponds to the no-slip boundary condition. At $\xi \to \infty$, formula (4) matches the drag for a spherical gas bubble. Solution (4) is written in the laboratory frame of reference, where volume-averaged velocity $w_\infty = \varphi v_\infty + (1-\varphi)u_\infty$ of the disperse medium is equal to zero at $\mathbf{r} \to \infty$. Here, v_∞ is the velocity of disperse particles. In the present case, $w_\infty \approx u_\infty$, since $\varphi \ll 1$. It is important to note that formula (4) characterizes correctly the influence of curvature of the body surface on the slip length [16].

Within the macroscopic (averaged over volume Ω) model of a gas bubble emulsion, parameter ξ is given and may only be determined experimentally. To calculate macroscopic parameter ξ, one needs first to solve the problem in its full formulation (with account for the collective hydrodynamic interaction of bubbles with the sphere surface) at the microstructural level (i.e., within the Ω volume) and then to average the solution. This problem has not yet been solved under mixed ($0 < \xi < \infty$) Navier boundary conditions (1) on the surface of a spherical body. However, the self-consistent field method, which is discussed in detail in [17,18] and briefly in the Appendix, made it possible to obtain an averaged analytical solution for two extreme cases [4]: the motion of a hard sphere ($\xi = 0$) and a spherical bubble ($\xi = \infty$). This solution takes the form

$$\mathbf{F} = -6\pi\eta_1 R\mathbf{V}\big[1 + k(\sigma)\varphi\big], \tag{5}$$

where $\sigma = a/R$, $k(\sigma) = k_s(\sigma)$ corresponds to the case of $\xi = 0$, and $k(\sigma) = k_b(\sigma)$ corresponds to $\xi = \infty$. Coefficients $k_s(\sigma)$ and $k_b(\sigma)$ were calculated with an accuracy of $O(\delta^7)$ with respect to small parameter $\delta = a/l$, where l is the average distance between the centers of neighboring disperse particles. Figure 1 presents the dependences of coefficients $k_s(\sigma)$ and $k_b(\sigma)$ on the ratio of characteristic sizes of the problem and the disperse medium.

Solutions (5) may be used to estimate the limits of variation of slip parameter ξ with σ and the bubble concentration: ξ_{up} (corresponds to dependence k_b) and ξ_{down} (corresponds to dependence k_s). To do this, one needs to equate Basset's solution (4) to the corresponding averaged solutions (5). The result is as follows:

$$\xi_{up} = \frac{\lambda_b}{R} = \frac{[1-k_b(\sigma)]\varphi}{1+[3k_b(\sigma)-2]\varphi},$$

$$\xi_{down} = \frac{\lambda_s}{R} = \frac{[1-k_s(\sigma)]\varphi}{1+[3k_s(\sigma)-2]\varphi}, \tag{6}$$

$$\frac{F_s}{F} = C = \frac{1+\varphi}{1+k(\sigma)\varphi} = \Big|_{\varphi\ll 1}$$

$$\approx 1 + \big[1-k(\sigma)\big]\varphi + O(\varphi^2). \tag{7}$$

It is evident from formula (7) that the relative error of determination of the drag force ($\Delta = 100(F_s/F - 1)\,\%$) does not exceed $\Delta \leqslant 100\varphi\,\%$ if the slip effect is neglected. Figure 2 shows the possible limits of variation of dimensionless slip

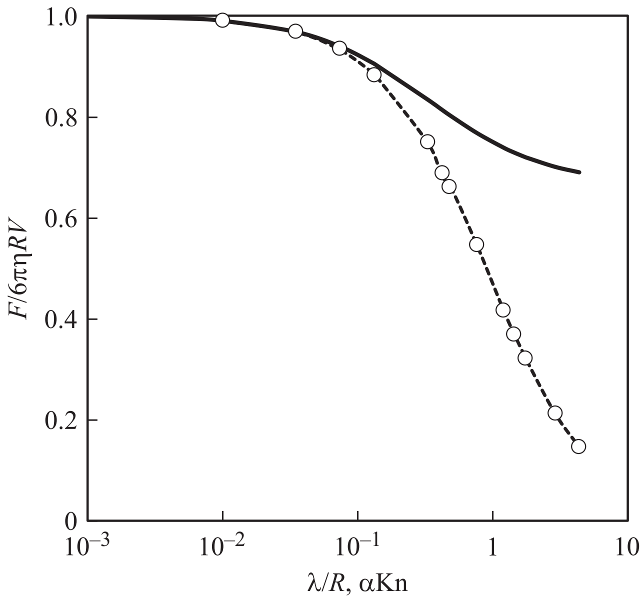

Figure 3. Comparison of the dependence of the dimensionless drag force on slip parameter ξ for a sphere moving in a bubble medium (solid curve) with experimental dependence $F_{exp}/F_s = f(\alpha\mathrm{Kn})$ in rarefied air (circles).

length ξ with parameter σ calculated using formulae (6) for different volume concentrations of bubbles.

Let us use correspondence (3) to compare (Fig. 3) theoretical dependence $F/F_s = f(\xi)$ for a bubble medium (7) with detailed measurements [19] of dependence $F_{exp}/F_s = f(\alpha\mathrm{Kn})$ of the drag force for spherical particles in a rarefied gas on the Knudsen number. It can be seen from Fig. 3 that the theoretical dependence matches the experimental one within the $0 < \alpha\mathrm{Kn} < 0.1$ $(0 < \xi < 0.1)$ range with an error less than 1 %. This is hardly surprising, since this range of Knudsen numbers is precisely the one in which a rarefied gas may be regarded as a continuum and its dynamics may be characterized by the Navier–Stokes equations [12]. At $\xi < 0.01$, the no-slip boundary conditions may be applied with an error less than 1 %, since $F = F_s$; within the $0.01 < \xi < 0.1$ range, one needs to use Navier slip boundary conditions (2). In addition, the above-mentioned agreement between theoretical and experimental data verifies the applicability of the $\boldsymbol{\tau}_s = \beta\mathbf{u}_w$ model boundary condition (1).

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Conflict of interest

The author declares that he has no conflict of interest.

Appendix. Brief description of the self-consistent field method

In the approximation of linear equations for a continuous carrier medium, the solution to the classical problem of motion of a spherical body (with radius R) in a liquid with an arbitrary finite number N of spheres (with radius a) is a superposition of perturbation fields from all spheres, each of which is expressed as a functional series containing tensor coefficients that are unknown in advance and determined by the specified boundary conditions on the surface of the corresponding sphere. Writing down the boundary conditions on the surface of an arbitrary sphere, one obtains a system of $N + 1$ equations for tensor coefficients corresponding to all spheres (including the large one). However, this system is incomplete, since it contains the velocities of dispersed particles that are unknown in advance. Therefore, a closing condition for mutual consistency of all perturbation fields is needed. Such conditions in the self-consistent field method are the equations of motion of spheres (Newton's second law) wherein the forces are expressed through the parameters of flow incident on a sphere: Faxén's and Beek's theorems are used for viscous [4,18] and inviscid [17] problems, respectively. The self-consistent field method allows one to calculate analytically all the quantities sought in the problem in any given approximation with respect to small parameter $\delta = a/l$.

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