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Opposition of two informational quantifiers of directional coupling between stochastic systems

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The work studies two widely known information-theoretic tools for estimation of directional couplings (mutual influences) between observed processes — transfer entropy and Liang–Kleeman information flow. They are formally similar, have measurement units with the same name and, indeed, often characterize a coupling in a similar sense. However, it is shown here with an exemplary stochastic system within the framework of dynamical causal effects that situations, where these quantifiers behave in an opposite way (one of them increases, while another one decreases) under a change of governing parameters, are typical. Then, the two quantifiers characterize a coupling in two essentially different senses which circumstance is important and should be taken into account in their practical applications.

Keywords: stochastic dynamical systems, information-theoretic quantifiers of directional couplings, time series.

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The problem of quantitative assessment of mutual influences (directional couplings) of two systems X and Y based on their observed time series $\{x_n\}$ and $\{y_n\}$, where x and y — state vectors of systems $x_n = x(t_n)$, $y_n = y(t_n)$, $t_n = n\Delta t$, Δt — sampling interval, and $n = 1, 2, \ldots, N$ — integer number, often arises in various physical and interdisciplinary studies. Several quantitative characteristics of such couplings (measures of „strength" of the influence) are used for this purpose (see, e.g., reviews [1–4]). Information-theoretical quantifiers are convenient (nonlinear) tools [1,2,5–19]. The most widely used among them are the transfer entropy (TE) [5] and the Liang–Kleeman information flow (LKIF) [6]. The TE concept gave rise to a huge number of studies (e.g., [1–4,7–9,11,15,17,19,20]) where various modifications of it [1,9,17] and methods of assessment based on time series [1,7,17] (specifically, physiological [2,12], financial [12], and climatic [3,9,17] ones) were proposed; even a dedicated monograph has already been published [12]. The LKIF concept has also been the subject of a number of studies ([4,10,13,14,16,18,19] are just a fraction of them) focused on the development of methods for its assessment based on time series [10,18], the introduction of various normalizations [10] and generalizations [16,18], and the application to climatic [13,14] and other data.

The formal similarity of these two quantifiers of couplings was noted immediately [6]. The units of their measurement are called the same: „bits" or "nats" [6,12]. It was also pointed out that TE and LKIF for the same coupling may assume different [6] (and even significantly different [4]) values. However, these quantifiers are generally expected to be similar in meaning and shift in the same direction (i.e., they should both decrease or both increase) when the parameters of the systems under study change. In the present study, we demonstrate that they, in contrast, may often shift in opposite directions.

The magnitude of TE in direction $Y \to X$ was defined [5] based on the concepts of data analysis for Markov processes as mutual information between the present state (y) and the future one (x) at a given present x. The $Y \to X$ LKIF magnitude was defined [6,10] on the basis of the evolutionary equation for the x distribution entropy. The component corresponding to the information flow from y to x was extracted from this equation by means of formal transformations. While their „conceptual origins" differ widely, TE and LKIF have found a consistent formulation and justification within the theoretical formalism of dynamic effects of directional coupling (DEDC) [4,11] in stochastic dynamic systems, which also revealed their difference [4].

The DEDC formalism defines the characteristic of directional coupling $Y \to X$ as a response of the future states of system X (called the X-response in [20]) to a change in parameters or initial states of system Y (called the Y-variation or „Y-wiggling" [20]). The existence of a stationary distribution of simultaneous states of two systems $\rho^{st}_{XY}(x, y) = \rho^{st}_X(x)\rho^{st}_{Y|X}(y|x)$ is assumed here. TE $T^{(t)}_{Y\to X}$ is then defined as the difference between the Shannon entropies of two ensembles of time realizations at $t > 0$ (response time) with one of them starting from the initial distribution $\rho^{*}_{XY}(x, y) = \delta(x - x_0)\delta(y - y_0)$, (i.e., the initial states of both systems x_0 and y_0 are given) and the other starting from $\rho^{**}_{XY}(x, y) = \delta(x - x_0)\rho^{st}_{Y|X}(y|x)$ (i.e., only the x_0 initial state is given). Then, $T^{(t)}_{Y\to X} = \left\langle H(p^{(t)}_X[x|\rho^{**}_{XY}]) - H(p^{(t)}_X[x|\rho^{*}_{XY}])\right\rangle_{\rho^{st}_{XY}(x_0,y_0)}$, where

$H(p) = -\int_{-\infty}^{\infty} p(x)\ln p(x)dx$ is the Shannon entropy of the p distribution, square brackets with a vertical bar indicate that $p_X^{(t)}$ is a „functional-conditional“ distribution for a given initial x and y distribution [4,20], and angle brackets denote averaging over x_0 and y_0 with weighting function $\rho_{XY}^{st}(x_0, y_0)$. An estimate of TE for a stationary time series may be obtained by searching for all values of x_n close to x_0 such that future $x_{n+t/\Delta t}$ will give an empirical estimate of the distribution $p_X^{(t)}[x|\rho_{XY}^{**}]$ and all pairs (x_n, y_n) close to (x_0, y_0) such that their future $x_{n+t/\Delta t}$ will give an empirical estimate of $p_X^{(t)}[x|\rho_{XY}^{*}]$ [1,5]. With the Gaussian approximation of the $p_X^{(t)}$ distributions, TE is estimated [15,17] as the difference between the logarithms of the x_t forecast error variance with and without account for y_0 [8].

LKIF turned out [4,19] to be equal to $L_{Y\to X}^{(t)} = \langle -\ln p_X^{(t)}[x|\rho_{XY}^{**}] + \ln p_X^{(t)}[x|\rho_{XY}^{*}]\rangle_{p_X^{(t)}[x|\tilde{\rho}_{XY}]\rho_Y^{st}(y_0)}$, where $\tilde{\rho}_{XY} = \rho_{X|Y}^{st}(x|y)\delta(y-y_0)$. This is the average difference of quantities of the form $-\ln p_X^{(t)}(x)$ for two ensembles starting from $\rho_{XY}^{*}(x,y) = \rho_X^{st}(x)\delta(y-y_0)$ and $\rho_{XY}^{**}(x,y) = \rho_X^{st}(x)\rho_{Y|X}^{st}(y|x)$, but averaged with such a weighting function that $L_{Y\to X}^{(t)}$ is not reduced to the difference in Shannon entropies. In contrast to TE, the $\rho_X^{st}(x)$ initial distribution is specified here instead of state x_0. In more exact terms, LKIF for continuous-time systems is the time derivative of response $l_{Y\to X} = \left.\frac{dL_{Y\to X}^{(t)}}{dt}\right|_{t=0}$. Formulae for its estimation based on time series are given in [10].

The dependences of TE and LKIF on parameters for the reference system with their analytical expressions available for it (which allows one to carry out an accurate analytical and numerical study without statistical evaluation) are examined below. These are two linear relaxators under the influence of white noise — a simple and widely used continuous-time stochastic model (two-dimensional Ornstein–Uhlenbeck process):

$$\dot{x} = -\alpha_X x + k_{XY} y + \xi_X(t),$$

$$\dot{y} = -\alpha_Y y + k_{YX} x + \xi_Y(t), \qquad (1)$$

where α_X, α_Y are the relaxation coefficients („speeds“ of systems); k_{XY}, k_{YX} are the coupling coefficients; and ξ_X, ξ_Y are mutually uncorrelated Gaussian white noises with zero mean and autocovariance functions $\langle\xi_X(t)\xi_X(t')\rangle = \Gamma_{XX}\delta(t-t')$ and $\langle\xi_Y(t)\xi_Y(t')\rangle = \Gamma_{YY}\delta(t-t')$. Differential quantities $\tau_{Y\to X} = \left.\frac{dT_{Y\to X}^{(t)}}{dt}\right|_{t=0}$ and $l_{Y\to X}$, which determine the role of coupling at small intervals t, were found analytically for system (1) in [4,10,15]. Their expressions are presented here in a convenient dimensionless form:

$$\tilde{T}_{Y\to X} \equiv \tau_{Y\to X}/\alpha_X = \beta_{XY}^2\left(\sigma_Y^2/\sigma_{Y,0}^2\right)\left(1 - r_{XY}^2\right)/4,$$

$$\tilde{L}_{Y\to X} \equiv l_{Y\to X}/\alpha_X = \beta_{XY} r_{XY}\left(\sigma_Y/\sigma_{Y,0}\right)/\left(\sigma_X/\sigma_{X,0}\right), \qquad (2)$$

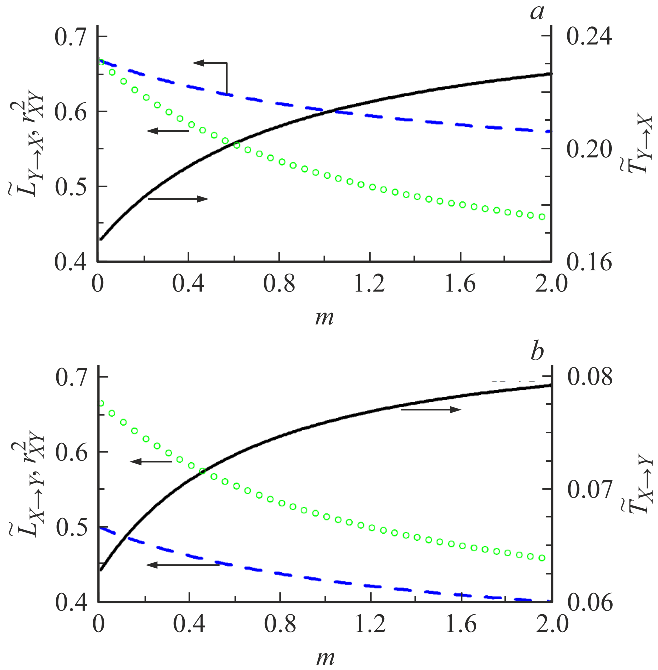

Figure 1. Quantifiers of directional couplings as functions of relative speed m of system Y for system (1) at $\beta_{XY} = 1, \beta_{YX} = 1/2$. a — in direction $Y \to X$; b — in direction $X \to Y$. Solid curves — dimensionless differential TE $\tilde{T}_{Y\to X}$, dashed curves — dimensionless LKIF $\tilde{L}_{Y\to X}$, and squares — correlation coefficient r_{XY} squared.

where $\sigma_{X,0}^2, \sigma_{Y,0}^2$ — variances of x and y under zero coupling, σ_X^2, σ_Y^2 — their variances under a given coupling with coefficients k_{XY}, k_{YX}, $\beta_{XY}^2 = k_{XY}^2\sigma_{Y,0}^2/\left(\alpha_X^2\sigma_{X,0}^2\right)$ — dimensionless $Y \to X$ coupling parameter, which is the ratio of two terms at the right-hand side of the first equation of system (1) by their dispersions in the „open-circuit“ mode, and r_{XY} — coefficient of correlation of x and y for distribution $\rho_{XY}^{st}(x,y)$. The expressions for reverse coupling $X \to Y$ assume a similar form with mutual substitution of subscripts X and Y. The values of r_{XY}, σ_X^2, and σ_Y^2 may be determined explicitly as a stationary solution to the equations of evolution of the second moments of distribution $\rho_{XY}(x,y)$ [11]:

$$\sigma_{X,0}^2 = \Gamma_{XX}/(2\alpha_X),$$

$$\sigma_X^2 = \sigma_{X,0}^2\left(1 + \frac{\alpha_X\beta_{XY}^2 + \alpha_Y\beta_{XY}\beta_{YX}}{(\alpha_X + \alpha_Y)(1 - \beta_{XY}\beta_{YX})}\right),$$

$$r_{XY} = \frac{\sigma_{X,0}\sigma_{Y,0}(\alpha_X\beta_{XY} + \alpha_Y\beta_{YX})}{\sigma_X\sigma_Y(\alpha_X + \alpha_Y)(1 - \beta_{XY}\beta_{YX})}$$

[15], so that

$$\beta_{XY}^2 = \frac{k_{XY}^2\Gamma_{YY}}{\alpha_X\alpha_Y\Gamma_{XX}},$$

$$\beta_{YX}^2 = \frac{k_{YX}^2\Gamma_{XX}}{\alpha_X\alpha_Y\Gamma_{YY}}.$$

The units of measurement of $\tilde{T}_{Y\to X}$ and $\tilde{L}_{Y\to X}$ depend on the logarithm base used in definitions: „nats“ (natural units)

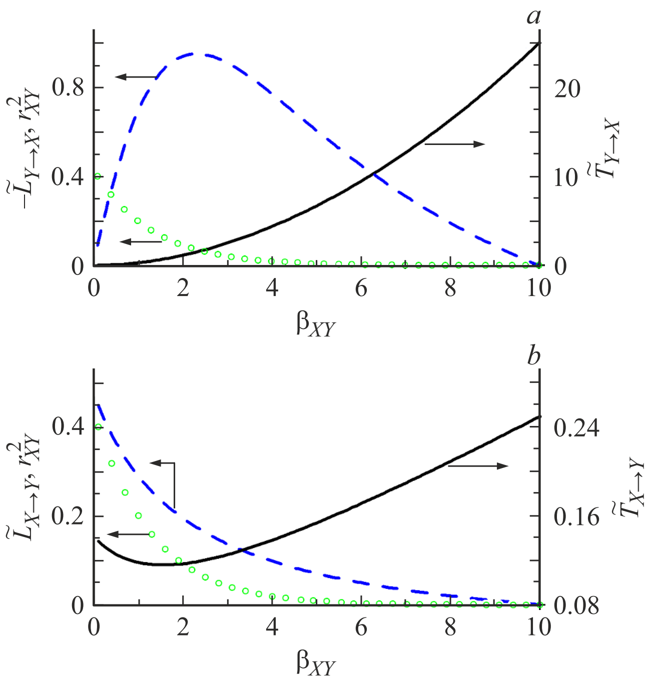

Figure 2. Coupling quantifiers for system (1) as functions of the dimensionless $Y \to X$ coupling parameter at $\beta_{YX} = -1$, $m = 10$. *a* — in direction $Y \to X$; *b* — in direction $X \to Y$. Solid curves — dimensionless differential TE $\tilde{T}_{Y\to X}$, dashed curves — dimensionless LKIF $\tilde{L}_{Y\to X}$ (taken with the opposite sign in panel *a*), and circles — correlation coefficient r_{XY} squared.

correspond to a natural logarithm (as in the present study), while bits correspond to a logarithm to base 2 [6,12].

Dimensionless differential quantifiers $\tilde{T}_{Y\to X}$ and $\tilde{L}_{Y\to X}$ (2) were calculated within a wide region of the space of dimensionless parameters of system (1). They often turn out to be similar to each other. Specifically, with unidirectional coupling $Y \to X$, they both grow quadratically in k_{XY} at small k_{XY}. In the case of bidirectional coupling with coefficients of the same sign, they again both grow as these coefficients increase, although, generally speaking, they are not proportional to each other. This similarity is quite expected (see also [4,15]), but certain cases of their significant difference are revealed below.

Let us fix the values of dimensionless coupling parameters $\beta_{XY} = 1$ and $\beta_{YX} = 1/2$. Their ratio $\beta_{XY}/\beta_{YX} > 1$. Therefore, the coupling is asymmetric in this sense, and the Y system is „leading," since the coupling parameter in the $Y \to X$ direction is larger. With constant β_{XY} and β_{YX}, we change dimensionless parameter $m = \alpha_Y/\alpha_X$, which is the relative relaxation rate of system Y. Let us, for example, fix noise intensities Γ_{XX}, Γ_{YY} and speed α_X; m is then adjusted by varying $\alpha_Y (m \propto \alpha_Y)$ (notably, $k^2_{XY} \propto \alpha_Y$ and $k^2_{YX} \propto \alpha_Y$ are also varied). Note that the dependences of quantifiers (2) on m also remain the same if all six dimensional parameters are adjusted in a different manner with the condition of constancy of β_{XY} and β_{YX} fulfilled. As m grows from $m < 1$ to $m > 1$, leading system Y, which was the slower of the two, becomes the faster one. The dimensionless differential TE then increases in both directions ($Y \to X$ and $X \to Y$; solid curves in Fig. 1), while the LKIF value decreases (dashed curves in Fig. 1). Specifically, as m increases from 0.1 to 2, the TE value increases by 20–30 %, while LKIF decreases by 15–20 % (Figs. 1, *a*, *b*). The pattern remains the same at other values of β_{XY} and β_{YX}; only the range of TE and LKIF values changes. The discovered phenomenon is typical for the space of dimensionless parameters of system (1). In particular, with β_{XY}, β_{YX} set, one free dimensionless parameter m remains, and the TE and LKIF values shift in opposite directions exactly when it changes.

It is interesting to note that TE and LKIF may shift in opposite directions even with a change in coupling parameter β_{XY} or β_{YX} (although it would seem that both quantifiers should increase with an increase in coupling coefficient): this occurs when positive β_{XY} changes at a fixed arbitrary m and negative β_{YX}. This case is illustrated in Fig. 2 with $m = 10$ and $\beta_{YX} = -1$: as β_{XY} increases from 2 to 10, the TE value in direction $Y \to X$, which has $|\beta_{XY}/\beta_{YX}| < m$ (i. e., the $Y \to X$ coupling is „relatively inferior" according to [15]), increases from 1 to 25 (in „nats"), while the LKIF value is negative and decreases in modulus from 1 to 0 (Fig. 2, *a*). In the opposite direction, TE increases from 0.12 to 0.24, while LKIF decreases from 0.2 to 0 (Fig. 2, *b*). These changes are very significant (several-fold or even order-of-magnitude). The results obtained at other fixed values of m and β_{YX} are qualitatively similar with a change in the range of TE and LKIF values.

In both highlighted cases, the opposite nature of TE and LKIF variation is attributable in part to a significant change in correlation coefficient r_{XY} (circles in Figs. 1, 2). It follows from expressions (2) that the values of $\tilde{T}_{Y\to X}$ and $|\tilde{L}_{Y\to X}|$ decrease and increase, respectively, as the modulus of this coefficient increases. In this case, either the direction or the rate of change of the x and y variances is inadequate for them to alter the directions of TE and LKIF variations that occur due to r_{XY}.

When TE and LKIF are estimated based on sufficiently long time series, the fluctuations of estimates are small, and they are close to the considered theoretical values of TE and LKIF. In the case of shorter series, one needs to take into account fluctuations in estimates and estimate confidence intervals (put forward a „null hypothesis"), which warrants a separate study. However, the effect of reduction/increase of the values of TE and LKIF under parameter variation should be preserved in general even for their estimates based on short series (if these estimates are still sufficiently accurate). The conclusion regarding the possible opposition of TE and LKIF should then also apply to such estimates.

Thus, it was demonstrated that two widely used information quantifiers of directional couplings between stochastic systems (TE and LKIF) may shift in opposite directions when the parameters of these systems change. Such scenarios were described explicitly and shown to be typical rather than degenerate in the parameter space. Since they occur in a simple stochastic system (1), such scenarios should be all the more common in more complex systems, including nonlinear ones. The discovered phenomenon of opposition of these two coupling quantifiers should then be typical in the practice of time series analysis ans should be taken into account when one interprets the results of assessment of couplings and makes conclusions as to whether the coupling intensifies and in what sense (according to which quantifier) does it intensify when certain conditions change. From a theoretical point of view, the obtained results demonstrate that the two considered information quantifiers of couplings and their units (be they nats or bits) may have completely different meanings. Therefore, the name „information flow," which is given to both of them [6,12] due to the use of an information-theoretical formalism in their definitions, does not in itself ensure their correct unambiguous interpretation. This shines a new light on the previously noted [4,11,20] non-triviality of the problem of „measuring the strength" of directional couplings.

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Conflict of interest

The author declares that he has no conflict of interest.

References

[1] K. Hlaváčková-Schindler, M. Paluš, M. Vejmelka, J. Bhattacharya, Phys. Rep., **441** (1), 1 (2007). DOI: 10.1016/j.physrep.2006.12.004
[2] M. Wibral, R. Vicente, J.T. Lizier, *Directed information measures in neuroscience* (Springer-Verlag, Berlin–Heidelberg, 2014). DOI: 10.1007/978-3-642-54474-3
[3] J. Runge, S. Bathiany, E. Bollt, G. Camps-Valls, D. Coumou, E. Deyle, C. Glymour, M. Kretschmer, M.D. Mahecha, J. Muñoz-Marí, E.H. van Nes, J. Peters, R. Quax, M. Reichstein, M. Scheffer, B. Schölkopf, P. Spirtes, G. Sugihara, J. Sun, K. Zhang, J. Zscheischler, Nat. Commun., **10**, 2553 (2019). DOI: 10.1038/s41467-019-10105-3
[4] D.A. Smirnov, Phys. Rev. E, **105** (3), 034209 (2022). DOI: 10.1103/PhysRevE.105.034209
[5] T. Schreiber, Phys. Rev. Lett., **85** (2), 461 (2000). DOI: 10.1103/PhysRevLett.85.461
[6] X.S. Liang, R. Kleeman, Phys. Rev. Lett., **95** (24), 244101 (2005). DOI: 10.1103/PhysRevLett.95.244101
[7] M. Vejmelka, M. Palus, Phys. Rev. E, **77** (2), 026214 (2008). DOI: 10.1103/PhysRevE.77.026214
[8] L. Barnett, A.B. Barrett, A.K. Set, Phys. Rev. Lett., **103** (23), 238701 (2009). DOI: 10.1103/PhysRevLett.103.238701
[9] J. Runge, J. Heitzig, V. Petoukhov, J. Kurths, Phys. Rev. Lett., **108** (25), 258701 (2012). DOI: 10.1103/PhysRevLett.108.258701
[10] X.S. Liang, Phys. Rev. E, **90** (5), 052150 (2014). DOI: 10.1103/PhysRevE.90.052150
[11] D.A. Smirnov, Phys. Rev. E, **90** (6), 062921 (2014). DOI: 10.1103/PhysRevE.90.062921
[12] T. Bossomaier, L. Barnett, M. Harre, J.T. Lizier, *An introduction to transfer entropy. Information flow in complex systems* (Springer, Cham, 2016). DOI: 10.1007/978-3-319-43222-9
[13] A. Stips, D. Macias, C. Coughlan, E. Garcia-Gorriz, X.S. Liang, Sci. Rep., **6**, 21691 (2016). DOI: 10.1038/srep21691
[14] D.F.T. Hagan, G. Wang, X.S. Liang, H.A.J. Dolman, J. Climate, **32** (21), 7521 (2019). DOI: 10.1175/JCLI-D-18-0881.1
[15] D.A. Smirnov, Phys. Rev. E, **102** (6), 062139 (2020). DOI: 10.1103/PhysRevE.102.062139
[16] X.S. Liang, Entropy, **23** (6), 679 (2021). DOI: 10.3390/e23060679
[17] R. Silini, C. Masoller, Sci. Rep., **11**, 8423 (2021). DOI: 10.1038/s41598-021-87818-3
[18] C. Pires, D. Docquier, S. Vannitsem, Physica D, **458**, 133988 (2024). DOI: 10.1016/j.physd.2023.133988
[19] D.A. Smirnov, Chaos, **34** (3), 033130 (2024). DOI: 10.1063/5.0189544
[20] D.A. Smirnov, Radiophys. Quantum Electron., **67**, 181 (2024). DOI: 10.1007/s11141-025-10364-9.

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