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Telescopic mode of a cathode lens

© A.A. Trubitsyn, E.Yu. Grachev, E.G. Kochergin

Ryazan State Radio Engineering University, Ryazan, Russia
E-mail: assur@bk.ru

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Means of modeling a cathode lens with an almost arbitrary configuration of electrodes in the paraxial approximation have been developed, and conditions for the implementation of the telescopic mode have been determined. The interrelation of the parameters providing this mode of operation of the lens has been studied. An electron-optical scheme has been developed that guarantees the telescopic mode of a cathode lens of a real (non-idealized) design.

Keywords: electronic optics, paraxial optics, emission system, cathode lens.

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The cathode of a cathode lens is immersed in an external field produced by the potentials of focusing electrodes (in most cases, the Wehnelt electrode [1]) and the anode. The cathode lens considered here is to be used in emission systems with their perveance lower than $10^{-2}\,\mu\mathrm{A/V}^{3/2}$ [2].

The parameters of the overwhelming majority of electron-optical systems may be predicted with the use of axially symmetric models with an accuracy sufficient for practical application. The fundamental equation of axially symmetric paraxial electron optics is a linear differential equation of the second order, the solution of which is particle trajectory $r = r(z)$. It takes the following form in cylindrical coordinate system $r0z$ [1]:

$$r'' + \frac{\Phi'}{2\Phi}r' + \frac{\Phi''}{4\Phi}r = 0, \qquad (1)$$

where $\Phi = \Phi(z)$ is the potential distribution on symmetry axis $0z$.

Unlike conventional types of electron lenses (immersion, single, aperture lens, etc.), a cathode lens presents difficulties in solving Eq. (1) that are associated with a mathematical singularity near the top of the cathode with coordinate $z = z_c$, since $\Phi_c = \Phi(z_c) = 0$ and $\Phi'_c \neq 0$. Here and elsewhere, subscript c denotes the values of all functions on the cathode surface.

The theory of solving differential equations with this type of singularity is complete (see, e.g., [3]): one of the particular solutions $p = p(z)$ of Eq. (1) is an analytical function, and the other solution $g = g(z)$ is expressed through analytical function $q = q(z)$

$$g = \sqrt{\Phi}q, \qquad (2)$$

and it follows from (1) and (2) that function $q = q(z)$ satisfies equation

$$q'' + \frac{3}{2}\frac{\Phi'}{\Phi}q' + \frac{3}{4}\frac{\Phi''}{\Phi}q = 0, \qquad (3)$$

and functions $p(z)$ and $q(z)$ are automatically subject to boundary conditions

$$p_c = q_c = 1, \qquad p'_c = q'_c = -\frac{\Phi''_c}{2\Phi'_c}. \qquad (4)$$

The trajectory equation is the general solution of a differential equation of the second order and, consequently, may be written as a sum of two linearly independent particular solutions

$$r(z) = a\,p(z) + bg(z),$$

where a and b are constants determined from the initial conditions on cathode surface $z = z_c$. Yakushev has obtained [4] the expressions for a and b for a cathode lens and trajectory equation

$$r(z) = r_c p(z) + \frac{2\sqrt{\varepsilon}}{\Phi'_c}\sin\vartheta_c g(z), \qquad (5)$$

where ε is the initial energy of electron escape from the cathode, which is regarded as a mathematical quantity of the second order of smallness; r_c is the coordinate of electron escape from the cathode surface; and ϑ_c is the initial motion angle.

The fascination of modern researchers with numerical methods and the overestimation of importance of accuracy levels achieved there have become an obstacle to the development and widespread use of the mathematical apparatus presented in [4] and the systematic analysis of the properties of cathode lenses. Therefore, cathode lenses have been little studied to date (not only within the aberration theory, but also in the paraxial approximation). The only example of a study offering new insights into a cathode lens is [5], where unique modes of cathode lenses were discovered by analyzing trajectory equation (5). At the same time, the results reported in [5] require elaboration and generalization, since they were obtained under the assumption of infinitely small gaps between electrodes and the axial distribution of potential $\Phi(z)$ was determined using the method of separation of variables.

The method allows one to calculate the potential distribution function only for idealized electron-optical systems with a simple boundary configuration. We develop an alternative (more general) approach where the axial distribution of potential $\Phi(z)$ is determined using the numerical boundary element method coupled with the proprietary technique for exact estimation of singular and quasi-singular integrands [6]. This approach provides an opportunity to simulate systems with electrodes of virtually arbitrary thickness and shape. The distribution function of axial potential $\Phi(z)$, which is calculated by the boundary element method at discrete nodes, is interpolated by splines. Well-known formulas for numerical differentiation are used to determine the values of derivatives $\Phi'(z)$, $\Phi''(z)$, $p'(z)$, and $g'(z)$.

In brief, the aim of the present study is to investigate cathode lenses of real (non-idealized) designs in the telescopic mode.

Before actually examining the properties of a cathode lens in the telescopic mode, we differentiate Eq. (5):

$$r'(z) = r_c p'(z) + \frac{2\sqrt{\varepsilon}}{\Phi'_c} \sin\vartheta_c g'(z). \tag{6}$$

It is the analysis of Eqs. (5) and (6) that reveals a number of unique cathode lens modes [5]. Specifically, the telescopic mode, wherein a parallel flux ($\sin\vartheta_c = 0$) of electrons leaving the cathode surface remains parallel $r'(z) = 0$ in the $z \geqslant z_{im}$ image space in the uniform field region, is defined by the system of expressions

$$\begin{cases} \sin\vartheta_c = 0, \\ p'(z) = 0, \quad z \geqslant z_{im}, \end{cases} \tag{7}$$

where z_{im} is the image space boundary.

The implementation of the telescopic mode in a paraxial cathode lens is ultimately determined by the specific type of potential distribution $\Phi(z)$ in (1) and (4) and does not depend on the initial values of energy ε and electron angle ϑ_c. In a three-electrode cathode lens of the kind shown in Fig. 1, *a*, which consists of disk-shaped cathode C and two cylindrical (intermediate and accelerating) electrodes of the same diameter d, this specific type of $\Phi(z)$ is determined by length l and potential V of the intermediate electrode at fixed potential V_{acc} of the accelerating electrode with its length being significantly larger than diameter d. The plots of particular solutions $p(z)$ and $g(z)$ for such a lens with $l/d = 0.7$ and $V/V_{acc} = 0.141$ are shown in Fig. 1, *b*. The section of the $p(z) = \text{const}$ dependence at $z \geqslant z_{im}$ is noteworthy as a characteristic feature of the telescopic mode where $z_{im} \approx d$.

The results of calculations for the studied cathode lens within the proposed approach, which relies on solving Eqs. (1) and (3), suggested a relation between length l and potential V of the intermediate electrode (Fig. 1, *c*) that enables telescopic mode (7).

The dependences shown in Fig. 1 were obtained in the paraxial approximation for small interelectrode gaps $\delta = 0.002d$ (i.e., in the $\delta \ll d$ case). A comparison of the data in Fig. 1, *c* and the results of independent studies [5] of the same lens performed under the assumption of infinitely small interelectrode gaps reveals that the relative difference between the corresponding $V/V_{acc} = f(l/d)$ dependences does not exceed 0.2 %, which provides indirect evidence of validity of our conclusions regarding the telescopic mode.

However, the question remains open about the conditions and limits of applicability of the paraxial approximation in studies of a cathode lens.

Let us note once again that the lack of influence of energy ε on the course of trajectories $r(z)$ in telescopic mode (7) follows from Eq. (5). At the same time, the numerical „reference" analysis of the studied lens in Focus Pro [7], where the electric field in the working area of the lens (and not only on the 0z axis) is determined numerically and the trajectories are found by time integration of the standard system of Newtonian equations of motion, makes it necessary to correct the statement made. The energy dependence may in fact be noticeable, but its influence decreases with decreasing relative energy ε/V_{acc} and may be considered insignificant below the level of 10^{-3}, which is illustrated by dependence $\beta = \beta(\varepsilon/V_{acc})$ in Fig. 2, *a*, where β is the angle of inclination of the straight section of the outermost electron trajectory to the 0z axis on crossing the $z = z_{out} = l + 2d$ plane. This refers to the trajectory with initial coordinate $r_c = d_{in}/2$ in the numerical experiment, where $d_{in} = 0.2d$ is the initial diameter of the electron flux emitted from the cathode. Let us recall once again that angle $\beta = \text{arctg}[r'(z_{out})] = 0$ within the entire energy range in the paraxial approximation. The above-mentioned level of relative energy $\varepsilon/V_{acc} = 10^{-3}$ does, first, specify the order of smallness of ε and, second, is characteristic of most known emission systems, such as electron microscopes and microfocus X-ray tubes; it is important that the paraxial approximation turns out to be suitable for these systems.

Initial angles $\vartheta_c = 0$ in the telescopic mode, but electrons emitted from the cathode in actual experiments are distributed according to the cosine law over the entire range of ϑ_c from -90 to $+90°$. The influence of the initial angular spread on the output parameters of the electron flux was investigated numerically (not in the paraxial approximation) in the Focus Pro environment [7]. The results of trajectory analysis (Fig. 2, *b*) within the range of polar angles ϑ_c from -70 to $+70°$, which was chosen in order to exclude uninformative „tails" near the -90 and $+90°$ boundaries of the full range, suggest that the initial angular spread has a significant influence on electron trajectories; however, the degree of influence decreases with decreasing relative energy ε/V_{acc}. It does indeed follow from the data in Fig. 2, *b* that the angular spread (for an arbitrary initial radial coordinate $r_c \leqslant d_{in}$) relative to the central horizontal trajectory $\vartheta_c = 0°$ at the exit of the system in plane $z = z_{out} = l + 2d$ decreases from $\pm 2.5°$ at $\varepsilon/V_{acc} = 10^{-3}$ to $\pm 0.07°$ at $\varepsilon/V_{acc} = 10^{-6}$. Thus, when electrons are

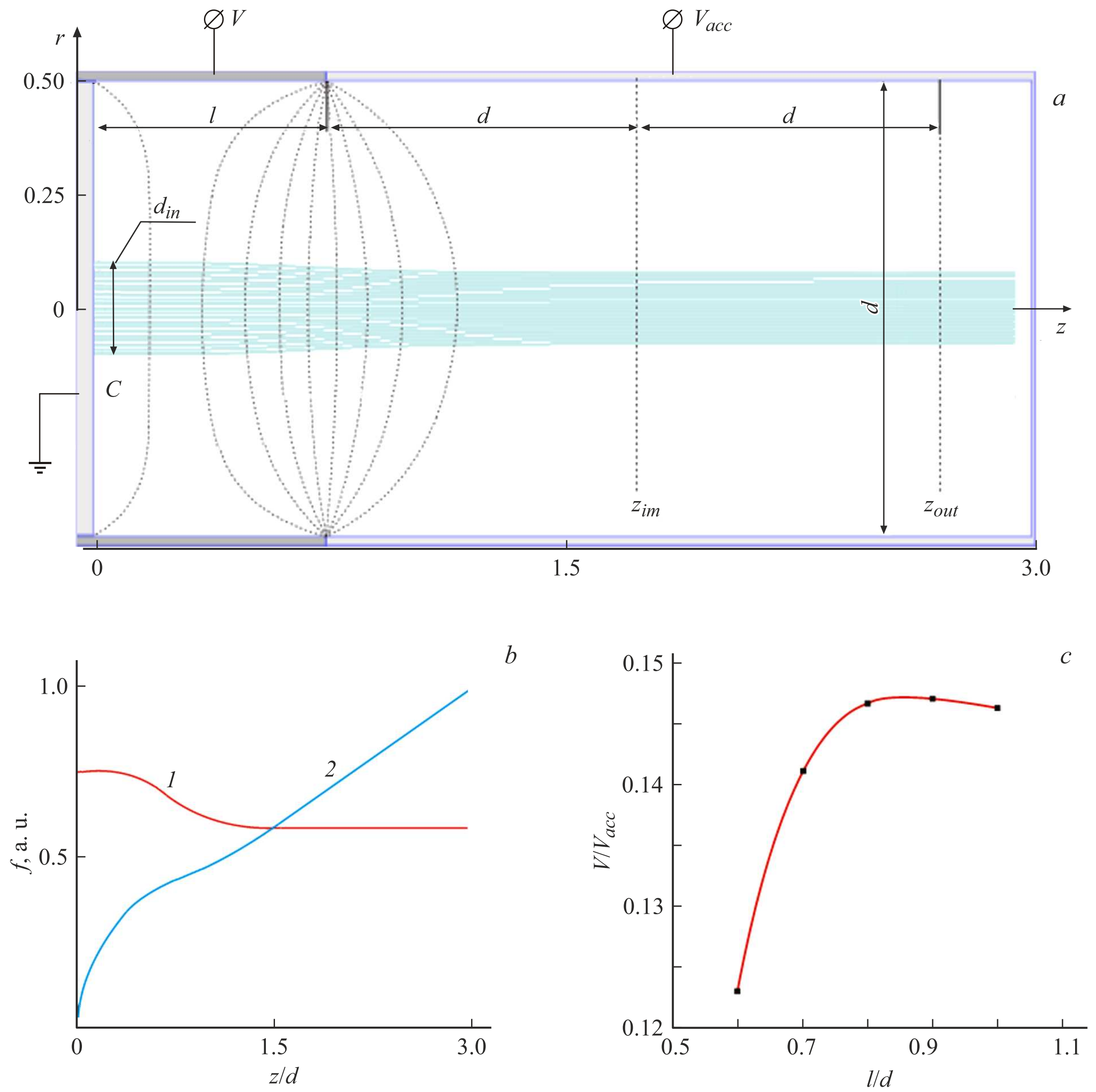

Figure 1. Modeling of a three-electrode cathode lens in the paraxial approximation. a — Trajectories (5) of electrons with $\vartheta_c = 0$ and $r_c = 0{-}0.2d$ in the telescopic mode; b — particular solutions: *1* — p, *2* — g; c — relation between potential V of the intermediate electrode and its length l that enables telescopic mode (7).

emitted from the cathode surface in a total solid angle of 2π sr in the telescopic mode of the cathode lens at low relative energies ($\varepsilon/V_{acc} < 10^{-4}$), an electron flux parallel to the optical axis of the lens with an angular spread of several tenths of a degree forms at the output.

The lens design examined above is not very practical, since the interelectrode gaps are small and cannot withstand high voltages on the order of several tens of kilovolts (or more) that are applied between the electrodes in typical emission systems. The approach to modeling cathode lenses developed in the present study is suitable for lenses with any electrode configuration, including those with wide interelectrode gaps. The lens shown schematically in Fig. 3 is an example of such an electrode system that provides a high level of electrical strength.

The search for conditions supporting the telescopic mode consists in repeating the following steps: set length l of the intermediate electrode and find such a potential V of this electrode that ensures the fulfillment of the second condition of system (7); notably, the axial distribution of potential used to find particular solution $p(z)$ from (1) is calculated by the boundary element method. Specifically, the results of calculations demonstrate that the telescopic mode at $l/d = 0.7$ (Fig. 3) is established by applying

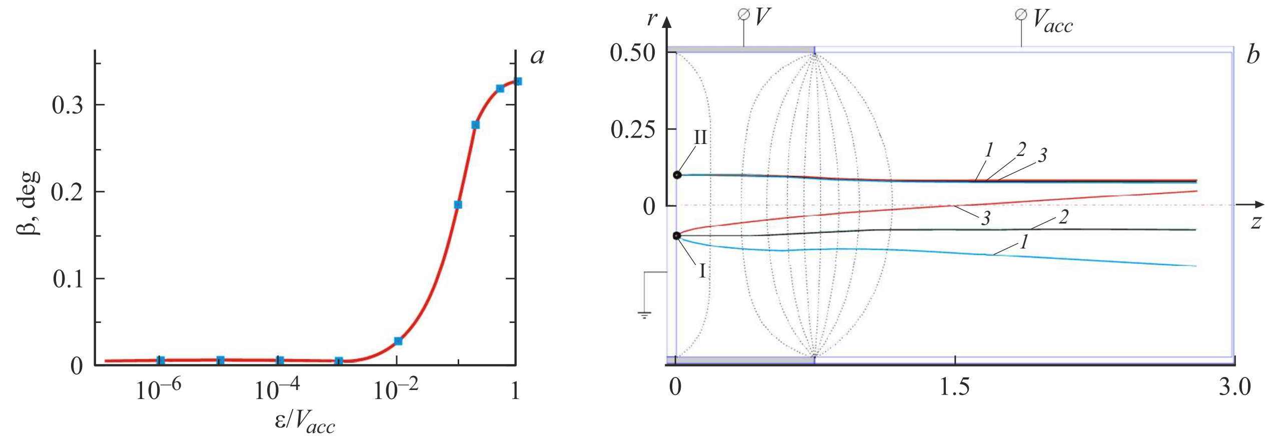

Figure 2. Effect of the initial energy and initial angles on the parallelism of trajectories at the exit of a cathode lens in the telescopic mode. *a* — Angular spread at $\vartheta_c = 0$; *b* — trajectories $\varepsilon/V_{acc} = 10^{-3}$ (I) and 10^{-6} (II), $\vartheta_c = -70$ (*1*), 0 (*2*) and +70° (*3*).

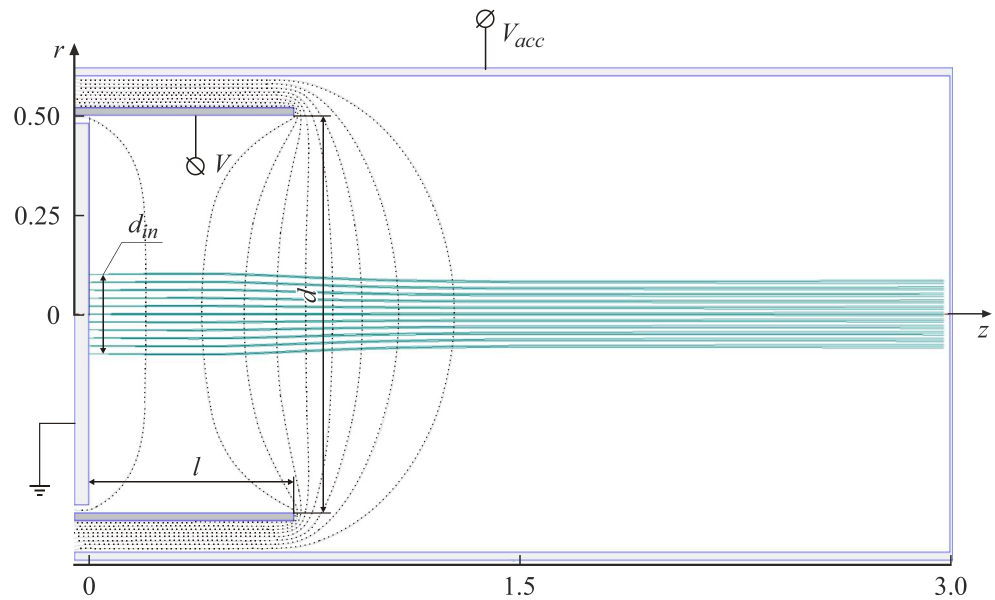

Figure 3. Results of numerical analysis of a cathode lens of a non-idealized design in the telescopic mode. $\varepsilon/V_{acc} = 10^{-6}$, $d_{in}/d = 0.2$, $r_c = 0{-}d_{in}$, and ϑ_c varies from −70 to +70°.

potential $V = 0.137V_{acc}$, which differs only slightly from the corresponding potential $V = 0.141V_{acc}$ of the idealized design (Fig. 1, *a*).

Trajectory analysis needs to be performed in order to examine in more detail the properties of the lens in the telescopic mode.

Figure 3 presents the results of numerical (not in the paraxial approximation) trajectory analysis of the lens within the range of angles ϑ_c from −70 to +70° with initial relative electron energy $\varepsilon = 10^{-6}V_{acc}$ and initial flux diameter $d_{in} = 0.2d$. The calculated data suggest that the electron flux maintains a high level of parallelism at the system output in plane $z = l + 2d$. The following conclusions may also be made:

— the angle of inclination of central trajectories corresponding to $\vartheta_c = 0$ increases with r_c, but does not exceed 0.02° at $r_c = d_{in}/2$;

— the total angular spread of the two outermost trajectories with $\vartheta_c = -70$ and +70° relative to the central trajectory does not exceed 0.17° regardless of r_c.

The telescopic mode will be useful in constructing emission systems with flat cathodes where electron fluxes need to be transported over long distances.

It should be noted that similar problems for triode-type electron beam guns forming electron fluxes of different spatial configurations (parallel ones included) were also solved in the optics of intense electron beams [2].

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Conflict of interest

The authors declare that they have no conflict of interest.

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