# Relaxation processes in a disordered Luttinger liquid

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The Luttinger liquid model, which describes interacting electrons in a single-channel quantum wire, is completely integrable in the absence of disorder and as such does not exhibit any relaxation to equilibrium. We consider relaxation processes induced by inelastic electron−electron interactions in a disordered Luttinger liquid, focusing on the equilibration rate and its essential differences from the electron−electron scattering rate as well as the rate of phase relaxation. In the first part of the paper, we review the basic concepts in the disordered Luttinger liquid at equilibrium. These include the elastic renormalization, dephasing, and interference-induced localization. In the second part, we formulate a conceptually important framework for systematically studying the nonequilibrium properties of the strongly correlated (non-Fermi) Luttinger liquid. We derive a coupled set of kinetic equations for the fermionic and bosonic distribution functions that describe the evolution of the nonequilibrium Luttinger liquid. Remarkably, the energy equilibration rate in the conducting disordered quantum wire (at sufficiently high temperature, when the localization effects are suppressed by dephasing) is shown to be of the order of the rate of elastic scattering off disorder, independent of the interaction constant and temperature.

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## 1. Introduction

Strongly correlated electron systems in one dimension (1d) have become an area of immense interest from the perspective of both fundamental and technological aspects of nanophysics. Intense experimental effort has focused on such realizations of quantum wires with a few or single conducting channels as cleaved-edge [1], V-groove [2], and crystallized-in-a-matrix [3] semiconductor quantum wires, coupled quantum Hall edges running in opposite directions [4,5], single-wall carbon nanotubes [6], polymer nanofibers [7], and metallic nanowires [8,9]. Central to much of the fascinating physics of the 1d systems is that electron−electron ($e-e$) interactions in 1d geometry can have dramatic effects leading to the emergence of a Luttinger liquid (LL) [10]. The latter constitutes a canonical example of a non-Fermi liquid, in which quasiparticle fermionic excitations are inappropriate to describe low-energy physics. At the foundation of the conventional LL theory is a description in terms of bosonic elementary excitations (plasmons, spinons) [10]. Following this approach, the ground-state properties of a clean LL are well understood for arbitrary strength of interaction. Much has also been learned about the LL in the presence of a single compact scatterer [10,11]. However, as far as a disordered LL is concerned, a number of important questions, even at the most fundamental level, remained largely unanswered until very recently (for a review see Ref. [12]).

In the presence of disorder, quantum interference of scattered electron waves leads to the effects of Anderson localization [13]. Similarly to $e-e$ interactions, the lower the dimensionality, the stronger the localization effects. In a 1d electron gas, all electron states are localized even for an arbitrarily weak random potential and the localization length is the mean free path. In the case of noninteracting electrons, the quantum localization in 1d has been studied in great detail (see, e.g., Ref. [14]). A principal complication that arises in the disordered LL is that the quantum interference phenomena yielding the Anderson localization are conventionally treated in terms of fermions, by employing the concepts of interference and dephasing of fermionic excitations. The question of to what extent the notion of phase relaxation in the localization problem is applicable to the (non-Fermi) LL is therefore of crucial importance. Recently, this problem was addressed in Refs. [12,15], where the interaction-induced dephasing rate that governs the localization term in the conductivity of the disordered LL was calculated.

Another conceptually nontrivial aspect of the interplay between disorder and interaction concerns the nonequilibrium properties of the LL. In the homogeneous case, the LL model is completely integrable and as such does not exhibit any relaxation to equilibrium: an arbitrary excited state will never decay to the equilibrium state characterized by temperature. Of central importance is therefore the question of how the equilibration of fermionic and/or bosonic excitations in the LL occurs in the presence of disorder.

This paper is primarily concerned with various relaxation processes associated with inelastic interactions between electrons in the disordered LL. Specifically, we focus on the rates of $e-e$ scattering, phase relaxation, and energy relaxation, with emphasis on the essential differences be-

tween them. In Sec. 2 we begin with the formulation of the model. Section 3 highlights a few aspects of temperature-dependent screening of disorder in 1d. Section 4 covers the problem of phase relaxation — this discussion largely follows the results of Refs. [12,15] and serves as the starting point for our approach to nonequilibrium physics of the LL. In Sec. 5 we consider energy relaxation and introduce a general framework [16] for studying the behavior of the disordered LL out of equilibrium.

## 2. Model

Let us specify the model. By decomposing the electron operator into right- and left-moving parts, $\psi(x) = \psi_+(x) + \psi_-(x)$, the Hamiltonian of a disordered LL is written as

$$H = H_{\text{kin}} + H_{\text{ee}} + H_{\text{dis}}, \tag{1}$$

$$H_{\text{kin}} = -v_F \sum_{\mu=\pm} \int dx \, \psi_\mu^\dagger \left( i\mu\partial_x + k_F \right) \psi_\mu, \tag{2}$$

$$H_{\text{ee}} = \frac{1}{2} \sum_{\mu=\pm} \int dx \left( n_\mu V_f n_{-\mu} + n_\mu \tilde{V}_f n_\mu \right), \tag{3}$$

$$H_{\text{dis}} = \int dx \left[ U_b(x) \psi_-^\dagger \psi_+ + \text{H.c.} \right]. \tag{4}$$

Here $n_\mu = \psi_\mu^\dagger \psi_\mu$ is the density of the right and left movers and their dispersion relation is linearized about two Fermi points at the wavevectors $\pm k_F$ with the velocity $v_F$. Throughout the paper we consider spinless electrons (for spin-related effects see Ref. [17]).

The $e-e$ interaction, Eq. (3), is characterized by the Fourier components of the short-range (screened) interaction potential with zero momentum transfer $V_f$ (forward scattering between right and left movers) and $\tilde{V}_f$ (forward scattering of electrons from the same chiral branch on each other). Unless the right and left movers are spatially separated (as in coupled quantum Hall edges), $\tilde{V}_f = V_f$. The Luttinger model *per se* does not include backward scattering characterized by the Fourier component $V_b$ with momentum transfer $\pm 2k_F$. For spinless electrons, however, $V_b$ can be trivially incorporated by shifting $V_f \to V_f - V_b$, since two types of scattering — due to $V_f$ and $V_b$ — are then related to each other as direct and exchange processes. The local interaction between identical fermions $\tilde{V}_f$ yields no scattering, but, due to a quantum anomaly in the LL model, generates a shift of the Fermi velocity $v_F \to v_F^* = v_F + \tilde{V}_f/2\pi$. It is customary to parametrize the strength of $e-e$ interaction by means of the Luttinger constant $K$:

$$K = \left( \frac{1-\alpha}{1+\alpha} \right)^{1/2}, \quad \alpha = \frac{V_f}{2\pi v_F^*}. \tag{5}$$

The velocity of elementary excitations (plasmons) in a clean LL is given by

$$u = v_F^* (1 - \alpha^2)^{1/2}, \tag{6}$$

which transforms for $\tilde{V}_f = V_f$ into $u = v_F/K$.

The low-energy theory described by the Hamiltonian (1) is only then well-defined when supplemented by an ultra-violet energy cutoff $\Lambda$. The latter depends on microscopic details of the problem and obeys

$$\Lambda = u/\pi\lambda, \tag{7}$$

where the length scale $\lambda$ is set by the lattice constant, the Fermi wavelength, or the spatial range of interaction in the original microscopic theory, whichever gives the smallest $\Lambda$. Thus the complete set of parameters defining the LL model in the absence of disorder includes $v_F^*$, $V_f$, and $\Lambda$. It is worth noting that the input parameters of the low-energy theory include Fermi-liquid-type renormalizations coming from energy scales larger than $\Lambda$; in particular, the „bare" $v_F$ in Eq. (1) in general is not an interaction-independent constant if the interaction is strong $(1 - K \sim 1)$.

The term $H_{\text{dis}}$, Eq. (4), describes backscattering of electrons off a static random potential $U(x)$. The latter is taken to be of white-noise type with the correlators of backscattering amplitudes

$$\overline{U_b(x)U_b^*(0)} = \overline{U(x)U(0)} = w\delta(x) \tag{8}$$

and $\overline{U_b(x)U_b(0)} = 0$. The forward-scattering amplitudes are omitted in Eq. (4) since they can be gauged out in the calculation of the conductivity.

## 3. Elastic scattering

One of the characteristic features of a LL is a large renormalization of the strength of disorder (8) by $e-e$ interaction. In particular, the conductivity without any localization [13] or pinning [18] effects included („Drude conductivity") is $\sigma_{\text{D}}(\omega, T) = e^2 v_F/\pi[-i\omega + M(\omega, T)]$, where the disorder-induced scattering rate in the dc limit

$$\frac{1}{\tau(T)} = \text{Re}\, M(0, T) = a_K \frac{1}{\tau_0} \left( \frac{\Lambda}{T} \right)^{2(1-K)} \tag{9}$$

grows as a power law with decreasing temperature $T$ for repulsive interaction ($K < 1$). The momentum relaxation rate in the absence of interaction is given by $\tau_0^{-1} = 2w v_F^{-1}$ with $w$ from Eq. (8). Calculating the Drude conductivity at finite $\omega$ and sending $\omega \to 0$ afterwards allows to unambiguously determine the coefficient [19] $a_K = \Gamma^2(1 + K)/\Gamma(2K)$ in the relaxation rate. Here and below the disorder is supposed to be weak in the sense that $\Lambda\tau \gg 1$.

The underlying physics of the renormalization (9) can be described in terms of the $T$-dependent screening of individual impurities; specifically, in terms of scattering by Friedel oscillations which slowly decay in real space and are cut off on the spatial scale of the thermal length. At this level, the only peculiarity of the LL as compared to higher dimensionalities is that the renormalization of $\tau$ is more singular and, most importantly from the calculational point of view, necessitates going beyond the Hartree-Fock approximation, even for weak interaction (see, e. g., Ref. [20]).

In general, not only the strength of disorder but also the strength of interaction is subject to renormalization and depends on $T$, so that the function $\tau(T)$ is not a simple power law. An important question, therefore, is under what condition the exponent in Eq. (9) is given by the bare interaction coupling constant. One of the approaches to the problem was formulated in Ref. [21] in terms of a bosonic renormalization group (RG). The RG approach does not allow to obtain the $K$-dependent prefactor $a_K$ in Eq. (9), but is particularly beneficial in predicting the $T$ dependence of the Drude conductivity. For spinless electrons, the one-loop RG equations read

$$dK/d\mathscr{L} = f(K)\mathscr{D}, \qquad (10)$$

$$d\mathscr{D}/d\mathscr{L} = (3 - 2K)\mathscr{D}, \qquad (11)$$

where $\mathscr{L} = \ln L/\lambda$ and $\mathscr{D} = 2w\lambda/\pi u^2$. For the Drude conductivity (i.e., as long as the localization effects are not included, see Sec. 4), the spatial scale $L$ is given by the thermal length $u/T$. The scattering rate $1/\tau(T)$ is then proportional to $T\mathscr{D}(T)$. The function $f(K) = -K^2/2 + (1 + K^2)(3 - 2K)/4$ vanishes at $K = 1$, so that interaction is not generated by disorder (in the original equations of Ref. [21], the coupling constant $K$ contains an admixture of disorder and therefore the corresponding $f(1) \neq 0$, see Ref. [12] for a discussion of this point); moreover, the interaction (hence $1 - K$) does not change sign in the course of renormalization.

The RG flow (10), (11) is characterized by a separatrix which behaves as $\mathscr{D} = 8(K - 3/2)^2/9$ for $K > 3/2$ and terminates at $K = 3/2$. For the bare (taken at $L = \lambda$) values of $\mathscr{D}$ and $K$ that lie below the separatrix (i.e., for the case of strong attractive interaction with $K > 3/2$), the disorder strength $\mathscr{D}$ renormalizes to zero, otherwise $\mathscr{D}$ grows with increasing $L$ to a strong-coupling point with $\mathscr{D} \sim 1$. The renormalization of the coupling constant $K$ by disorder is essential if the RG trajectory is close to the parabola $\mathscr{D} = 8(K - 3/2)^2/9$. For example, if the RG flow passes through the point $K = 3/2$, the integration of Eqs. (10) and (11) gives for $\mathscr{D} \ll 1$

$$\mathscr{D} - \mathscr{D}_0 = \mathscr{D}_0 \tan^2\left(\frac{3\mathscr{D}_0^{1/2}}{2^{3/2}} \ln \frac{L}{L_0}\right), \qquad (12)$$

where $\mathscr{D}_0$ and $L_0$ are the values of $\mathscr{D}$ and $L$ at $K = 3/2$ and the sign of $\ln(L/L_0)$ is positive for running $K < 3/2$ and negative otherwise. One sees that $\mathscr{D}$ grows with increasing $L$ for $K < 3/2$ as

$$\mathscr{D} = 8/9 \ln^2(l/L) \qquad (13)$$

(for $\mathscr{D}_0 \ll \mathscr{D} \ll 1$). Here the renormalized mean free path $l$ (the scale at which $\mathscr{D} \sim 1$) obeys $\ln(l/L_0) = 2^{1/2}\pi/3\mathscr{D}_0^{1/2}$. The logarithmic dependence of $\mathscr{D}$ on $L$ is precisely due to the renormalization of $K$.

On the other hand, if the bare $K < 3/2$, the RG trajectory follows Eq. (12) with $L_0 = \lambda$ and $\mathscr{D}_0$ understood as the

bare value of $\mathscr{D}$ only at $\mathscr{D} - \mathscr{D}_0 \gg (K - 3/2)^2$. Integrating Eqs. (10), (11) in the opposite limit

$$\mathscr{D} - \mathscr{D}_0 \ll (K - 3/2)^2, \qquad (14)$$

one gets

$$\mathscr{D} = \mathscr{D}_0(L/\lambda)^{3-2K}, \qquad (15)$$

which corresponds to Eq. (9). Equation (14) thus answers the question of when the renormalization of $K$ may be neglected. Notice that for repulsive interaction ($K < 1$) the condition (14) is satisfied for the whole range of $\mathscr{D} \ll 1$ (which is where the RG equations (10), (11) are valid). It follows that for the most relevant case of direct Coulomb interaction the renormalization of interaction on ballistic scales ($\mathscr{D} \ll 1$) plays no role and the exponent in Eq. (9) is $T$-independent and given by the bare value of $K$ (the one in a clean system). In other words, the renormalization of disorder for repulsive interaction reduces to the renormalization of an individual impurity. It is worth emphasizing that this does not mean that the disorder-induced correction to the bare value of $1 - K$ is small: in fact, the correction is of the order of $1 - K$ itself when $\mathscr{D} \sim 1$. The point is that the exponent of $\mathscr{D}(L)$ and, correspondingly, of the renormalized scattering rate $1/\tau(T)$ is not given by the running coupling constant $K$, but rather is accumulated on the whole RG trajectory.

## 4. Phase relaxation

The renormalization of $\tau$ stops with decreasing $T$ at

$$T\tau(T) \sim 1, \qquad (16)$$

since the long-range Friedel oscillations created by disorder are cut off even at zero $T$ on the spatial scale of the disorder-induced mean free path. This condition gives the zero-$T$ mean free path $l \propto \tau_0^{1/(3-2K)}$ (notice that Eq. (16) is also expressible as $\mathscr{D}(L) \sim 1$ with $L = u/T$) and, correspondingly, the zero-$T$ localization length $\xi \sim l$. It is important to stress, however, that the above condition does not correctly predict the onset of localization with decreasing $T$ — in contrast to the argument, frequently stated in the literature (see, e.g., Ref. [10] and references therein) and based on the RG equations (13), (14), which treat scalings with the length scales $L$ and $u/T$ as interchangeable. While substituting $u/T$ for $L$ is justified for the „elastic renormalization" (Eq. (9)), the one-loop equations (13), (14) miss, by construction, the interference effects (coherent scattering on several impurities) that lead to localization. The status of the RG [21] is thus that of the Drude formula for interacting electrons. The $T$ dependence of the conductivity $\sigma(T)$, however, comes not only from the $T$-dependent screening of disorder (Eq. (9)), but also from the localization term in $\sigma(T)$ whose amplitude is governed by phase relaxation due to inelastic $e-e$ scattering. The temperature below which the localization effects become strong is, in contrast

to Eq. (16), determined by the condition

$$\tau(T)/\tau_\phi(T) \sim 1, \qquad (17)$$

where $\tau_\phi$ is the weak-localization dephasing time. Notice that for weak interaction $(1 - K \approx \alpha \ll 1)$, Eq. (17) is satisfied at much higher $T$ than Eq. (16). Below we introduce the notion of dephasing of localization effects in the disordered LL and analyze the phase relaxation in the limit of weak interaction.

The very applicability of the notion of dephasing, as we know it from the studies of higher-dimensional Fermi-liquid systems, to the LL is not altogether apparent. A subtle question concerns the nature of elementary excitations in the LL, especially in the presence of disorder. The clean LL is a completely integrable model which is represented in terms of noninteracting (hence nondecaying) bosons; however, the phase relaxation in electron systems is conventionally described in terms of interacting fermions. Physically, the difficulty is related to the fact that the bosonized approach describes propagation of density fluctuations, whereas the natural language for quantum interference phenomena is that of quantum amplitudes. To study the interference effects and their dephasing, one has therefore to either proceed with the standard bosonization, poorly suited to describe the quantum interference in the inhomogeneous case, or try to define the observables in such a way that they can be expressible in terms of decaying fermionic excitations. In what follows in this section, we take the latter path and give a succinct analysis of the phase relaxation in the disordered LL, based on the results obtained within the „functional bosonization" formalism [12] and the quasiclassical formalism [15], both of which combine the fermionic and bosonic approaches to the problem.

Let us first point out one of the subtleties of the LL model, which is crucial to our discussion of the phase and energy relaxation. The Golden rule expression for the $e-e$ collision rate at equilibrium, as follows from the Boltzmann kinetic equation, reads

$$\frac{1}{\tau_{ee}(\epsilon)} = \int d\omega \int d\epsilon' \, \mathcal{K}(\omega)$$
$$\times \left( f^h_{\epsilon-\omega} f_{\epsilon'} f^h_{\epsilon'+\omega} + f_{\epsilon-\omega} f^h_{\epsilon'} f_{\epsilon'+\omega} \right), \qquad (18)$$

where $f_\epsilon$ is the Fermi distribution function and $f^h_\epsilon = 1 - f_\epsilon$. Consider the *clean* case. Then the scattering kernel $\mathcal{K}(\omega) = \mathcal{K}^H_{++}(\omega) + \mathcal{K}^H_{+-}(\omega) + \mathcal{K}^F(\omega)$ to second order in the interaction is given by

$$\mathcal{K}^H_{++} = \frac{\tilde{V}^2_f}{\pi^3 \rho} \int \frac{dq}{2\pi} \left[ \mathrm{Re} D_+(\omega, q) \right]^2, \qquad (19)$$

$$\mathcal{K}^H_{+-} = \frac{V^2_f}{\pi^3 \rho} \int \frac{dq}{2\pi} \mathrm{Re} D_+(\omega, q) \mathrm{Re} D_-(\omega, q), \qquad (20)$$

and $\mathcal{K}^F = -\mathcal{K}^H_{++}$. The Hartree terms $\mathcal{K}^H_{++}$ and $\mathcal{K}^H_{+-}$ are related to scattering of two electrons from the same or different chiral spectral branches, respectively, $\mathcal{K}^F$ is the Fock

(exchange) counterpart of $\mathcal{K}^H_{++}$, the thermodynamic density of states $\rho = 1/\pi v_F$, and $D_\pm = i\pi\rho/(\omega \mp qv_F + i0)$ are the two-particle propagators in the clean limit.

Substituting Eqs. (19), (20) in Eq. (18), we obtain the lowest-order result for the $e-e$ scattering rate at the Fermi level $(\epsilon = 0)$ in terms of the corresponding contributions to the retarded electronic self-energy $\Sigma_+$ defined by $G^R_+(\epsilon, p) = [\epsilon - v_F p - \Sigma_+(\epsilon, p)]^{-1}$, where $G^R_+$ is the retarded Green's function for right-movers. Specifically, $\tau^{-1}_{ee} = -2\mathrm{Im}\Sigma_+(0, 0)$ with $\Sigma_+(0, 0) = \Sigma^H_{++} + \Sigma^H_{+-} + \Sigma^F$, where

$$\mathrm{Im}\Sigma^H_{+\pm} = -\frac{\pi}{2}\alpha^2 v_F \int d\omega\, \omega \left( \coth \frac{\omega}{2T} - \tanh \frac{\omega}{2T} \right)$$
$$\times \int dq\, \delta(\omega - v_F q)\delta(\omega \mp v_F q), \qquad (21)$$

$\Sigma^F = -\Sigma^H_{++}$, and we put $V_f = \tilde{V}_f$. One sees that the contribution of $\Sigma^H_{++}$ is diverging. For spinless electrons, however, the divergency is canceled by the exchange interaction. Indeed, as we have discussed in Sec. 2, the $\tilde{V}_f$ interaction drops out of the problem in this case, inducing only a shift of the velocity $v_F \to v^*_F$. It is worth noting that the „Hartree-Fock cancellation" is only exact for the point-like interaction (when $\tilde{V}_f$ is independent of the transferred momentum), otherwise $\tilde{V}_f$ yields a nonzero contribution [22] to $\tau^{-1}_{ee}$. The latter is small and can be neglected in the low-$T$ limit for $\tilde{V}_f = V_f$ but is the only one present for $V_f = 0$, which is the case, e.g., for an isolated quantum Hall edge. The remaining term $\Sigma^H_{+-}$ gives

$$\tau^{-1}_{ee} = -2\mathrm{Im}\Sigma^H_{+-} = \pi\alpha^2 T. \qquad (22)$$

This may look very similar to the familiar $T^2$ or $T^2 \ln T$ dependence of the $e-e$ scattering rate in clean three- or two-dimensional electron systems, respectively. However, the nontrivial point — which demonstrates the peculiarity of the LL model — is that $\tau^{-1}_{ee}$ in Eq. (22) is determined by

$$\omega, q = 0, \qquad (23)$$

i.e., by scattering processes with infinitesimally small energy transfers, in contrast to higher dimensions where the characteristic transfer is of order $T$. On the other hand, it is worth emphasizing that $T\tau_{ee} \gg 1$ for $\alpha \ll 1$, which in Fermi-liquid theory is commonly referred to as one of the conditions for the existence of a Fermi liquid. In this respect, the weakly interacting LL, while being a canonical example of a non-Fermi liquid, reveals the typical Fermi-liquid property. The LL physics (e.g., the power-law singularity of the tunneling density of states at the Fermi level) is in fact encoded in the singular *real* part of the self-energy $\Sigma_+(\epsilon, p)$ (for more details see Ref. [12]).

It is the property (23) that actually makes the 1d case special as far as the dephasing problem is concerned. Indeed, in the spirit of Ref. [23], soft inelastic scattering with $qv_F, \omega \ll \tau^{-1}_\phi$ is not expected to contribute to the dephasing of the localization effects. In higher dimensions,

in the ballistic regime $T\tau \gg 1$, this infrared cutoff is of no importance and the dephasing rate $\tau_\phi^{-1}$ is given [24] by $\tau_{ee}^{-1}$. However, in view of Eq. (23), $\tau_\phi^{-1}$ in 1d cannot possibly reduce to $\tau_{ee}^{-1}$.

The dephasing rate $\tau_\phi^{-1}$ can be accurately defined by calculating the weak-localization correction to the conductivity of the disordered LL as a function of $T$ [12,15]. The leading localization correction $\Delta\sigma$ in the ballistic limit $\tau_\phi/\tau \ll 1$ is related to the interference of electrons scattered by three impurities. One of the diagrams contributing to $\Delta\sigma$ (for the complete set of diagrams at the leading order see Ref. [12,15]) is given by a „three-impurity Cooperon" (Fig. 1), which describes the propagation of two electron waves along the path connecting three impurities („minimal loop") in time-reversed directions. In the absence of dephasing, quantum interference processes involving a larger number of impurities sum up to exactly cancel (similarly to the noninteracting case [14]) the Drude conductivity $\sigma_D = e^2 v_F \tau/\pi$, where $\tau$ is given by Eq. (9). For $\tau_\phi/\tau \ll 1$, they only yield subleading corrections through a systematic expansion in powers of $\tau_\phi/\tau$.

Within the functional-bosonization description of the LL [25,26], extended in Ref. [12] to treat disordered systems, the interaction can be exactly accounted for by performing a local gauge transformation $\psi_\mu(x, \tau) \rightarrow \psi_\mu(x, \tau) \exp[i\theta_\mu(x, \tau)]$, where the bosonic field $\theta_\mu(x, \tau)$ is related to the Hubbard−Stratonovich decoupling field $\varphi(x, \tau)$ by

$$(\partial_\tau - i\mu v_F \partial_x)\theta_\mu(x, \tau) = \varphi(x, \tau). \quad (24)$$

Here $\tau$ is the Matsubara time. The correlator $\langle\varphi(x, \tau)\varphi(0, 0)\rangle = V(x, \tau)$ gives the dynamically screened interaction, for which the random-phase approximation (RPA) in the LL model is exact [27]. In the presence of impurities, the interaction can thus be completely gauged out to the backscattering impurity vertices — Eq. (24) is then *exact* for any given realization of the impurity potential. In Fig. 1, the fluctuating disorder-induced gauge factors are denoted by the wavy lines attached to the backscattering vertices: each impurity vertex contributes the factor $\exp[\pm(\theta_+ - \theta_-)]$ and the averaging over fluctuations of $\varphi$ pairs all the fields $\theta_\pm$ with each other. The interaction thus induces the factor

$$\exp(-S_C) = \langle\exp[i(\theta_f - \theta_b)]\rangle \quad (25)$$

to the Cooperon loop, where $\theta_f$ and $\theta_b$ are the phases accumulated by an electron propagating along the „forward" and „backward" paths and the averaging is performed over the fluctuations of the field $\varphi$. Notice that the averaging couples with each other not only the phases $\theta_\pm$ attached to the impurities shown in Fig. 1 but also those attached to impurities which yield damping of the dynamically screened interaction. As shown in Refs. [12,15], the boson damping is crucially important for the dephasing (see below) and a parametrically accurate approximation is to include impurity-induced backscattering in the effective
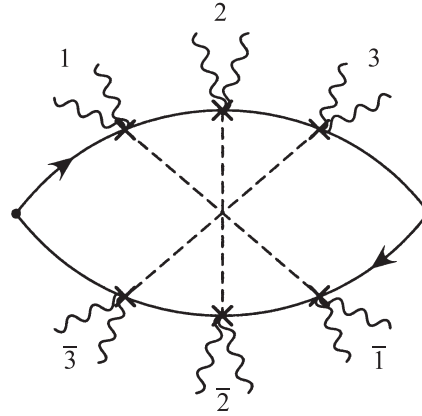


**Figure 1.** Three-impurity Cooperon diagram with interaction effects encoded in the fluctuating factors $\exp(\pm\theta_\mu)$ (denoted by the wavy lines) attached to the backscattering vertices (marked by the crosses). The dashed lines connect the backscattering vertices belonging to the same impurity (e.g., 1 and $\bar{1}$ refer to two backscatterings off impurity 1 at two different times).

interaction at the level of the disorder-dressed RPA („dirty RPA"). The total Cooperon action

$$S_C = S + S_{\text{renorm}} \quad (26)$$

accounts then for both the dephasing $(S)$ and the elastic renormalization of impurities $(S_{\text{renorm}})$ and we refer the reader for technical details of the formalism to Ref. [12].

The leading localization correction to the conductivity can be represented in the form [12,15]

$$\Delta\sigma = -2\sigma_D \int_0^\infty dt_c \int_0^\infty dt_a P_2(t_c, t_a) \exp[-S(t_c, t_a)], \quad (27)$$

where $P_2(t_c, t_a) = (1/8\tau^2)\exp(-t_c/2\tau)\Theta(t_c - 2t_a)$ is the probability density of return to point $x = 0$ after two reflections at points $x = ut_a$ and $x = -u(t_c/2 - t_a)$. Here $ut_c$ gives the total length of the Cooperon loop and $ut_a$, being the distance between two rightmost impurities, parametrizes the geometry of the loop. The classical trajectory for the Cooperon is characterized by a *single* velocity [12] and this is $u$ (the difference between $u$ and $v_F$ can be ignored for $\alpha \ll 1$, but uniformly on the whole trajectory). The phase relaxation is encoded in the dephasing action $S$ in Eq. (27), which is a growing function of the size of the Cooperon loop and cuts off the localization correction at $t_c \sim \tau_\phi$. The dephasing rate $\tau_\phi^{-1}$ is thus defined by the characteristic scale of $t_c$ on which the dephasing action $S \sim 1$.

In the limit $t_c \ll \tau$ the action reads [12,15]:

$$S(t_c, t_a) = 2\pi\alpha^2 T t_a (t_c - 2t_a)/\tau. \quad (28)$$

Substitution of Eq. (28) into Eq. (27) gives

$$\Delta\sigma = -\frac{1}{4}\sigma_D \left(\frac{\tau_\phi}{\tau}\right)^2 \ln\frac{\tau}{\tau_\phi} \propto \frac{1}{\alpha^2 T} \ln(\alpha^2 T), \quad (29)$$

where

$$\tau_\phi^{-1} = \alpha(\pi T/\tau)^{1/2}. \qquad (30)$$

One sees that the phase relaxation in the disordered LL occurs on time scales much longer than the lifetime $\tau_{ee}$:

$$\tau_\phi = (\tau_{ee}\tau)^{1/2} \gg \tau_{ee}. \qquad (31)$$

Note that $\tau_\phi^{-1}$ vanishes in the clean limit, in contrast to the total $e-e$ scattering rate — in agreement with the observation (23) and the basic fact [23] that scattering with energy transfers smaller than $\tau_\phi^{-1}$ is not effective in dephasing the localization effects.

The vanishing of the dephasing action at $\tau^{-1} \to 0$ can be made more transparent from the technical point of view by representing $S$ as a difference between the self-energy ($S_{\mathrm{ff}} + S_{\mathrm{bb}}$) and vertex ($S_{\mathrm{fb}} + S_{\mathrm{bf}}$) contributions. Here the terms $S_{ij}$, associated with an inelastic interaction between electrons propagating along the paths $x_i(t)$ and $x_j(t)$, where $i, j = $ (f) and (b) stand for the „forward" and „backward" time-reversed paths in the Cooperon, are given by

$$S_{ij} = -T \int \frac{d\omega}{2\pi} \int \frac{dq}{2\pi} \int_0^{t_c} dt_1 \int_0^{t_c} dt_2 \frac{1}{\omega} \mathrm{Im} V_{\mu\nu}(\omega, q)$$

$$\times \exp\{iq\,[x_i(t_1) - x_j(t_2)] - i\omega(t_1 - t_2)\}. \qquad (32)$$

Equation (32) is similar to that in higher dimensionalities („AAK action" [23]) with one subtle and important distinction. Because of the Hartree−Fock cancellation of the bare interaction $\tilde V_f$ between electrons from the same chiral branch (recall the discussion after Eq. (21)), the dynamically screened retarded interaction $V_{\mu\nu}(\omega, q)$ acquires the indices $\mu, \nu$ denoting the direction of motion of the interacting electrons: $\mu = \mathrm{sgn}\dot x_i$, $\nu = \mathrm{sgn}\dot x_j$. If one would keep both $V_f$ and $\tilde V_f$ processes in $V(\omega, q)$, the dephasing action in 1d could *not* be written in the form of Eq. (32) — in contrast to higher dimensionalities, where $S_{ij}$ is given by Eq. (32) with the „full" $V(\omega, q)$.

Neglecting the disorder-induced damping of the dynamically screened interaction yields

$$S_{\mathrm{ff}} = S_{\mathrm{fb}} = t_c/2\tau_{ee} \qquad (33)$$

and the exact cancellation of the self-energy and vertex parts in the total dephasing action, hence the vanishing of $\tau_\phi^{-1}$ (30) in the clean limit. It is thus only because of the small difference between $S_{\mathrm{ff}}$ and $S_{\mathrm{fb}}$ produced by the dressing of $V_{\mu\nu}(\omega, q)$ by impurities („dirty RPA" [12,15]) that the dephasing action (28) is not zero. The characteristic energy transfer $\omega$ in the processes that lead to the dephasing (i.e., contribute to the difference $S_{\mathrm{ff}} - S_{\mathrm{fb}}$) is much larger than $\tau^{-1}$ (more accurately, $\omega$ is spread over the range between $\tau_\phi^{-1}$ and $\tau^{-1}$, because of the logarithmic factor in Eq. (29)), which justifies the expansion of $S$ in powers of $\tau^{-1}$, while the condition $T\tau_\phi \gg 1$ justifies the quasiclassical treatment of the electromagnetic fluctuations in Eq. (32). Substituting Eq. (30) into Eq. (17) gives the temperature scale $T_1 \sim 1/\alpha^2\tau$ below which the localization effects become strong (for the behavior of the conductivity at $T \ll T_1$ see Ref. [28]). Note that $T_1\tau \gg 1$ for weak interaction.

## 5. Energy relaxation

We now turn to the nonequilibrium properties of the disordered LL [16]. Here we are primarily interested in the equilibration rate at which an excited state relaxes to equilibrium (other aspects of the nonequilibrium relaxation will be discussed elsewhere [16]). As mentioned in Sec. 1, the integrability of the clean LL model precludes energy relaxation. The absence of inelastic scattering in the LL deserves additional comment. For scattering of electrons from different chiral branches on each other, the energy and momentum conservation laws for linear electronic dispersion lead to two equalities: $\omega - v_F q = 0$ and $\omega + v_F q = 0$. These combine to give $\omega, q = 0$ and thus no energy exchange (cf. Eq. (23)). For scattering of electrons of the same chirality $\mu$, the energy-momentum conservation laws give a single equation $\omega - \mu v_F q = 0$ and at first glance the energy relaxation is allowed. Moreover, the relaxation might seem to be very strong since the Golden-rule expression for the probability of scattering contains the delta function $\delta(\omega - \mu v_F)$ squared. For the point-like interaction, the diverging Hartree and exchange terms cancel each other; however, for a finite-range interaction—despite the LL model being still completely integrable—the cancellation is no longer exact. The energy relaxation, nonetheless, is absent in the LL model for a generic shape of the interaction potential. The point is that beyond the Golden rule the diverging terms sum up to produce the dynamically screened interaction between electrons (exactly given by the RPA), which propagates with velocity $u(q) \neq v_F$. As a result, the probability of scattering contains a product of two delta functions $\delta(\omega - \mu v_F q)\delta[\omega - \mu u(q)q]$ with *different* velocities, which yields $\omega, q = 0$ for electrons from the same chiral branch as well.

The energy relaxation is thus only present if one goes beyond the clean LL model. One possibility comes from three-particle scattering [29], which occurs for a nonzero range of interaction provided that the electronic dispersion is nonlinear. The three-particle collision rate is small in the parameter $T/\epsilon_F \ll 1$, where $\epsilon_F$ is the Fermi energy. Another possibility is to take into account impurity backscattering, which may lead to a much stronger mechanism of energy relaxation.

It is important that the nonequilibrium state of the LL in general cannot be described in terms of a single — either bosonic or fermionic — distribution function. The simplest example to illustrate this point is that of the clean LL in which the left and right movers, separately at equilibrium within themselves, are characterized by the Fermi distribution functions $f_\epsilon^\pm = f_F(\epsilon - \mu_\pm)$ with different chemical potentials. Then the distribution functions $N^\pm(\omega)$ of the left and right plasmon modes are constructed as the convolutions of the fermion functions:

$$N^\pm(\omega) = \frac{1}{\omega} \int d\epsilon f_\epsilon^\pm(1 - f_{\epsilon-\omega}^\pm) = N_B(\omega), \qquad (34)$$

i.e., are given by the *equilibrium* Bose distribution, independent of $\mu_\pm$. This observation shows that the purely

bosonic description of the clean LL at a finite bias voltage is not complete. Such a „partial nonequilibrium" setup, in which the bosons are still at equilibrium, has been studied previously by employing the conventional bosonization (see, e.g., Ref. [30]). Furthermore, the nonequilibrium transport through a *single* impurity between *equilibrium* leads shifted by the voltage $\mu_+ - \mu_-$ has been studied in Ref. [31]. However, the standard scheme of bosonization will break down if the nonequilibrium distributions of the injected right- and left-moving electrons are not the Fermi distributions.

The challenge is thus to formulate a theoretical framework to describe a genuinely nonequilibrium situation in which both the bosonic and fermionic excitations are out of equilibrium. It is worth stressing that in the inhomogeneous case the nonequilibrium distribution functions do not obey the simple local relation (34), since the distribution functions of plasmons and electrons evolve with different velocities ($u$ and $v_F$). Notice also that the necessity of introducing both the bosonic and fermionic distribution functions is not peculiar to 1d: for higher-dimensional systems see Ref. [32].

Our approach to nonequilibrium phenomena in the LL uses as a base the formalism of the „functional bosonization", developed in Ref. [12] for the treatment of disordered LL at equilibrium. A conceptually similar formalism has been applied earlier for higher-dimensional disordered conductors in the study of the single-particle density of states out of equilibrium in Ref. [33]. The nonequilibrium tunneling density of states in the clean LL in Refs. [33,34]). We formulate the theory of the disordered LL out of equilibrium, which builds on the approaches of Refs. [35,36] and Ref. [32], in terms of the effective nonequilibrium real-time action. To account for the $e-e$ interaction, we introduce a dynamical field $\phi(x,t)$ which decouples the four-fermion term in the action by means of the conventional Hubbard−Stratonovich transformation.

The central object of our theory is the quasiclassical Green's function $\hat{g}(x, t_1, t_2)$ for electrons in the LL, taken at coinciding spatial points [37]:

$$\hat{g}(x, t_1, t_2) = \lim_{x' \to x} \Big[ 2i v_F g(x, x', t_1, t_2) \\ - \text{sign}(x - x') \delta(t_1 - t_2) \Big]. \quad (35)$$

This function, which is a $4 \times 4$ matrix in the Keldysh and channel (right/left) space, satisfies the Eilenberger equation [38]

$$i v_F \partial_x \hat{g} + [(i\partial_t \hat{\tau}_z - \hat{H}), \hat{g}] = 0, \quad (36)$$

where

$$\hat{H} = \hat{\phi} \hat{\tau}_z + \frac{1}{2} (U_b \hat{\tau}^+ + U_b^* \hat{\tau}^-). \quad (37)$$

Here and throughout this section below, $v_F$ means the renormalized velocity $v_F^*$ (see the discussion around Eq. (5)), so that the difference between $u$ and $v_F$ is of order $\alpha^2$ for small $\alpha$. Equation (36) describes the motion

of an electron in the random potential characterized by the backscattering amplitude $U_b(x)$ ([Eq. (8)) in the presence of the dynamic field $\hat{\phi}(x, t) = \text{diag}(\hat{\phi}^+, \hat{\phi}^-)$, where $\hat{\phi}^\mu(x, t) = \phi_1^\mu(x, t) + \hat{\sigma}_x \phi_2^\mu(x, t)$, $\phi_1^\mu$ and $\phi_2^\mu$ are the classical and quantum components of the Hubbard−Stratonovich field with chirality $\mu$, respectively. The Pauli matrices $\tau_z$ and $\tau^\pm = \tau_x \pm i\tau_y$ act in the channel space. We also introduce the Pauli matrices $\hat{\sigma}_{x,y,z}$ that act in the Keldysh space. The Hamiltonian $\hat{H}$ (37) is defined on the direct product of the time, Keldysh, and channel spaces. Accordingly, the commutator [ , ] in Eq. (36) is understood with respect to all three („discretized" time, Keldysh, and channel) indices. The operator $\partial_t$ acts as $\overrightarrow{\partial}_{t_1}$ from the left and as $(-\overleftarrow{\partial}_{t_2})$ from the right. For the case of linear electronic dispersion, assumed in the LL model, the Eilenberger equation (36) is *exact* for any given realization of the backscattering amplitude $U_b(x)$.

The next step is to average Eq. (36) over disorder. At this point we disregard the localization effects [12,15], which limits the applicability of the subsequent derivation to sufficiently high (effective) temperatures; specifically, for the length of the quantum wire larger than the mean free path to $T \gg T_1 \sim 1/\alpha^2\tau$ (see the end of Sec. 4). Under this condition we can perform the disorder averaging at the level of the self-consistent Born approximation, which gives

$$i\mu v_F \partial_x \overline{g^\mu} + \left[ i\partial_t - \hat{\phi} + \frac{i}{4\tau} \overline{g^{-\mu}}, \overline{g^\mu} \right] = 0 \quad (38)$$

for the disorder-averaged Green's function $\overline{g^\mu}$. In what follows we only deal with the averaged propagators and therefore omit the bar for brevity.

The Green's function $\hat{g}$ satisfies the normalization condition

$$\hat{g} \circ \hat{g} = \hat{1}\delta(t_1 - t_2), \quad (39)$$

where the dot denotes the convolution in all three (time, Keldysh, and channel) spaces. The constraint (39) enables us to formulate the effective action that reproduces Eq. (38) as its saddle point in the form essentially combining the actions derived in Refs. [35] and [36]:

$$S[\hat{g}, \hat{\phi}, \hat{A}] = - \frac{1}{2v_F} \text{Tr} \left[ (i\partial_t - \hat{\phi})\hat{\tau}_z + v_F \hat{A} \right] \hat{g} \\ - \frac{i}{2} \text{Tr} \hat{g}_0 T^{-1} \partial_x T - \frac{i}{8v_F\tau} \text{Tr} \hat{g}^+ \hat{g}^-. \quad (40)$$

The Green's function in Eq. (40) is represented as $\hat{g} = T\hat{g}_0 T^{-1} = \text{diag}(\hat{g}^+, -\hat{g}^-)$, where $\hat{g}_0 = \text{diag}(\hat{g}_0^+, -\hat{g}_0^-)$ corresponds to the saddle point of the action of the noninteracting problem and the unitary transformation $T$ (diagonal in the channel space) parametrizes possible fluctuations around $g_0$ (satisfying the constraint (39)), induced by fluctuations of the field $\hat{\phi}(x, t)$. To generate the response functions in the Keldysh formalism [39,40], we have also added the external-source term $\hat{A}(x, t) = a_1(x, t) + \hat{\sigma}_x a_2(x, t)$. The trace operation includes the summation over the Keldysh and space indices and the integration over time.

The Keldysh partition function of the system can now be expressed as a functional integral over $\hat{\phi}$,

$$\mathcal{Z}[A] \sim \int \mathcal{D}\phi^{\mu}_{1,2}(x,t)$$

$$\times \exp\left\{ iS[\hat{\phi},\hat{g},\hat{A}] + \frac{i}{2}\mathrm{Tr}\hat{\phi}\left(V_f^{-1}\hat{\tau}_x + \frac{1}{2\pi v_F}\right)\hat{\sigma}_x\hat{\phi}\right\}, \quad (41)$$

where $\hat{g}(x,t_1,t_2;[\hat{\phi}(x,t)])$ is understood as minimizing the action (40) for a given $\phi(x,t)$ under the constraint (39).

Having written the Eilenberger equation (38) and its action (40) we now use the standard technique [39,40] to derive the quantum kinetic equations. We proceed at one-loop order with respect to the effective interaction, which is equivalent to the „dirty RPA" [12]. The one-loop derivation is controlled by the small parameters $1/T\tau_{\phi} \ll 1$ and $\alpha \ll 1$. More specifically, following the framework of Ref. [32], we introduce three different distribution functions for each $\mu$. The first one, $f^{\mu}(\epsilon,x,t)$, describes the *bare* electrons, moving with velocity $v_F$. The other two, $N_p^{\mu}(\omega,x,t)$ and $N_g^{\mu}(\omega,x,t)$, describe two types of bosons, having velocities $u_p = v_F/K$ and $u_g = v_F$. The bosons of the first kind represent the usual plasmons ($p$) of the LL, whereas those of the second kind are „ghosts" ($g$) constructed from the bare electron–hole pairs, thus preventing from a double-counting of the degrees of freedom in the system (see the discussion around Eqs. (82)–(86) below).

We first apply the gauge transformation

$$\tilde{g}^{\mu}(x,t_1,t_2) = e^{-i\theta^{\mu}(x,t_1)}\hat{g}^{\mu}(x,t_1,t_2)e^{i\theta^{\mu}(x,t_2)}, \quad (42)$$

where $\hat{\theta}^{\mu} = \theta_1^{\mu} + \hat{\sigma}_x\theta_2^{\mu}$ has the same Keldysh structure as the field $\hat{\phi}$. This transformation is similar to that in Refs. [26,41], but different in that the equation of motion for the phase $\hat{\theta}$ in the field $\hat{\phi}$ will incorporate disorder, see Eq. (51) below. The „rotated" Green's functions $\tilde{g}^{\mu}$ are expressed in terms of the electron distributions $f_{\epsilon}^{\mu}(x,t)$, written in the time domain, as

$$\tilde{g}^{\mu} = \begin{bmatrix} \delta(t_1-t_2) & 2h^{\mu}(t_1,t_2,x) \\ 0 & -\delta(t_1-t_2) \end{bmatrix}, \quad (43)$$

where $h^{\mu} = \delta(t_1-t_2) - 2f^{\mu}(t_1,t_2,x)$,

$$f^{\mu}(t_1,t_2,x) = \int \frac{d\epsilon}{2\pi} e^{i\epsilon(t_1-t_2)} f_{\epsilon}^{\mu}[x,(t_1+t_2)/2], \quad (44)$$

and we impose the condition

$$f^{\mu}(t_1,t_2,x)\Big|_{t_1\to t_2} \to \frac{i}{2\pi(t_1-t_2+i0)}. \quad (45)$$

Equation (45) means that the fast charge and current fluctuations are now encoded in the fluctuations of the phase factors $e^{\pm i\hat{\theta}}$ in Eq. (42). The gauge-transformed action reads

$$S[\hat{\theta},\hat{\phi},\tilde{g}] = S_e + S_b + S_{\mathrm{int}} + S_{\mathrm{imp}}, \quad (46)$$

$$S_e = -\frac{1}{2v_F}\mathrm{Tr}(i\partial_t - \hat{L}_0\hat{\theta} - \hat{\phi})\hat{\tau}_z\tilde{g} - \frac{i}{2}\mathrm{Tr}\hat{g}_0 T^{-1}\partial_x T, \quad (47)$$

$$S_b = \frac{1}{2\pi v_F}\mathrm{Tr}\left[\frac{1}{2}(\partial_t\hat{\theta})\hat{L}_0\hat{\sigma}_x\hat{\theta} + \hat{\phi}\hat{\sigma}_x\partial_t\hat{\theta}\right], \quad (48)$$

$$S_{\mathrm{int}} = \frac{1}{2}\mathrm{Tr}\hat{\phi}\left(V_f^{-1}\hat{\tau}_x + \frac{1}{2\pi v_F}\right)\hat{\sigma}_x\hat{\phi}, \quad (49)$$

$$S_{\mathrm{imp}} = -\frac{i}{8v_F\tau}\mathrm{Tr}e^{-i(\hat{\theta}^- - \hat{\theta}^+)}\tilde{g}^+ e^{i(\hat{\theta}^- - \hat{\theta}^+)}\tilde{g}^-, \quad (50)$$

where $\hat{L}_0 = \partial_t + \hat{\tau}_z v_F\partial_x$.

We treat the fluctuations of $\hat{\theta}$ and $\hat{\phi}$ in the Gaussian approximation by expanding Eq. (46) around the saddle point of $S$ for a given $\hat{g}^{\mu}$. Optimizing then the action with respect to $\hat{\theta}$ for a given $\hat{\phi}$, we get a linear relation between $\hat{\theta}$ and $\hat{\phi}$:

$$\hat{D}_g^{-1}\theta = -\hat{\sigma}_x\phi, \quad (51)$$

where we introduce the vector notation $\theta = (\theta_1^+, \theta_1^-, \theta_2^+, \theta_2^-)^T$, $\phi = (\phi_1^+, \phi_1^-, \phi_2^+, \phi_2^-)^T$, $T$ stands for transposition, and the particle–hole propagator $D_g$ is constructed as

$$\hat{D}_g^{-1} = (\partial_t + \hat{\tau}_z v_F\partial_x)\hat{\sigma}_x + \frac{1}{2}\hat{\gamma}(1-\tau_x) \quad (52)$$

with

$$\hat{\gamma} = \frac{1}{\tau}\begin{pmatrix} 0 & -1 \\ 1 & 2B_{\omega} \end{pmatrix}, \quad (53)$$

$$B_{\omega} = \frac{1}{2\omega}\sum_{\mu}\int d\epsilon(1 - h_{\epsilon}^{\mu}h_{\epsilon-\omega}^{-\mu}). \quad (54)$$

The scattering operator $\hat{\gamma}$ (53) describes the decay/recombination of the collective electron–hole excitation into/from the electron and hole moving in opposite directions, assisted by impurity scattering. Note that the approximation (51) is equivalent[2] to the „dirty RPA" [12] in Sec. 4.

Substituting Eq. (51) back into the approximate quadratic action, we obtain the „dirty-RPA" propagator of the effective interaction

$$\langle\phi\phi^T\rangle = \frac{i}{2}\hat{V} = \frac{i}{2}(\hat{\sigma}_x\hat{\tau}_x V_f^{-1} - \hat{\Pi})^{-1}, \quad (55)$$

where

$$\hat{\Pi} = \frac{1}{2\pi v_F}\left[\hat{\sigma}_x(\partial_t\hat{D}_g)\hat{\sigma}_x - \hat{\sigma}_x\right] \quad (56)$$

is the polarization operator. By combining Eqs. (55) and (51) we get the correlator of the phases $\theta$ (cf. Ref. [42])

$$\langle\theta\theta^T\rangle = \frac{i}{2}\hat{D}_g\sigma_x\hat{V}\sigma_x\hat{D}_g = -\frac{i\pi v_F}{\partial_t}(\hat{D}_p - \hat{D}_g), \quad (57)$$

[2] Note that in Refs. [12,15] the propagator $\hat{D}_g$ in Eqs. (51) and (57) does not contain disorder (Eq. (24)) and reduces to $L_0^{-1}$, which is, however, sufficient for the calculation of $\tau_{\phi}$.

where $\hat{D}_p$ is the renormalized particle−hole propagator corresponding to the plasmon modes with velocity $u$ given by Eq. (6):

$$\hat{D}_p^{-1} = \left( \frac{\partial_t}{1 + \alpha\hat{\tau}_x} + v_F \hat{\tau}_z \partial_x \right) \hat{\sigma}_x + \frac{1}{2} \hat{\gamma}(1 - \hat{\tau}_x). \quad (58)$$

As follows from Eq. (50), the only phase that is coupled to the electron backscattering off disorder is $\Phi = \frac{1}{2}(\theta^- - \theta^+)$. Another observation is that the propagators of the fluctuations of $\theta$ have two different types of poles: $\omega = \pm uq$ and $\omega = \pm v_F q$, both smeared by disorder. It is thus convenient to define the correlator of the phase $\Phi$ as a difference of two terms:

$$\langle \Phi\Phi^T \rangle = \frac{i}{2} \left( \hat{\mathscr{L}}_p - \hat{\mathscr{L}}_g \right), \quad (59)$$

where

$$\hat{\mathscr{L}}_b = -\frac{i\pi v_F}{2\partial_t} \sum_{\mu\nu} \mu\nu \hat{D}_b^{\mu\nu} \quad (60)$$

and $b = p, g$ denotes the plasmon and ghost modes, which differ from each other in that the plasmon mode is characterized by velocity $u$, whereas the ghost mode by velocity $v_F$. Then the Wigner-transform of the Keldysh correlator $\langle \Phi\Phi^T \rangle_K$ has to be described by four different distribution function, $N_p^\pm(\omega, x, t)$ and $N_g^\pm(\omega, x, t)$, evolving with velocities $u$ and $v_F$ to the right and to the left:

$$\langle \Phi\Phi^T \rangle_K(\omega, q \approx \pm\omega/u, x, t)$$
$$= \left[ 2N_p^\pm(\omega, x, t) + 1 \right] \mathrm{Im}\,\mathscr{L}_p^A(\omega, q), \quad (61)$$

$$\langle \Phi\Phi^T \rangle_K(\omega, q \approx \pm\omega/v_F, x, t)$$
$$= - \left[ 2N_g^\pm(\omega, x, t) + 1 \right] \mathrm{Im}\,\mathscr{L}_g^A(\omega, q). \quad (62)$$

To derive the kinetic equation for the electron distribution function, the next step is to write down the equation of motion for the gauge-transformed Green's function $\tilde{g}^\mu$ (42). The latter follows from the relation

$$\frac{\delta}{\delta\tilde{g}^\mu} (S_e + S_{\mathrm{imp}}) = 0. \quad (63)$$

Using Eq. (51) we represent $S_e$ as

$$S_e = -\frac{1}{2v_F} \mathrm{Tr}(i\partial_t \hat{\tau}_z - \tilde{\phi})\tilde{g}, \quad (64)$$

where the shifted phase $\tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2\hat{\sigma}_x$,

$$\tilde{\phi}_\alpha = \sum_\beta (\hat{\sigma}_x \hat{\gamma})_{\alpha\beta} \Phi_\beta, \quad (65)$$

and $\hat{\gamma}$ is given by Eq. (53). Notice that the field $\tilde{\phi}$ does not depend on the chiral index $\mu$. The Eilenberger equation for $\tilde{g}^\mu$ thus reads

$$i\mu v_F \partial_x \tilde{g}^\mu + \left[ i\partial_t - \mu\tilde{\phi} + \frac{i}{4\tau} e^{2i\mu\hat{\Phi}} \tilde{g}^{-\mu} e^{-2i\mu\hat{\Phi}}, \tilde{g}^\mu \right] = 0. \quad (66)$$

Equation (66) has to be averaged over the fluctuations of the phase $\Phi$ with the correlator given by Eq. (59). Within the „dirty-RPA" it is sufficient to represent $\tilde{g}^\mu$ as a sum $\tilde{g}^\mu = \langle\tilde{g}^\mu\rangle + \delta\tilde{g}^\mu$, where $\langle\tilde{g}^\mu\rangle$ is the mean value, and take

into account only the term in $\delta\tilde{g}^\mu$ that is linear in the fluctuations of $\Phi$, keeping in mind that the quadratic-in-$\Phi$ fluctuations of $\tilde{g}^\mu$ are incorporated in the mean value.

By linearizing Eq. (66) around the average $\langle\tilde{g}^\mu\rangle$, we obtain $\delta\tilde{g}^\mu = -2(\delta f^\mu)\hat{\sigma}_+$, where the fluctuation of the distribution function obeys

$$\sum_\mu \left[ \hat{D}_{g,R}^{-1}(\omega) \right]^{\nu\mu} \delta f^\mu(\epsilon_1, \epsilon_2) = \frac{i\nu}{\tau} \sum_{\alpha=1,2} \lambda_\alpha^\nu(\epsilon_1, \epsilon_2)\Phi_\alpha(\omega). \quad (67)$$

In Eq. (67), $\delta f^\mu(\epsilon_1, \epsilon_2)$ is the Fourier transform of $\delta f^\mu(t_1, t_2, x)$ and $\omega = \epsilon_1 - \epsilon_2$. The source terms $\lambda_\alpha^\nu$ are expressed through the averages $h_\epsilon^\mu$ as

$$\lambda_1^\nu(\epsilon_1, \epsilon_2) = \frac{\nu}{2} \sum_\mu \mu(h_{\epsilon_2}^\mu - h_{\epsilon_1}^\mu), \quad (68)$$

$$\lambda_2^\nu(\epsilon_1, \epsilon_2) = 1 + B_\omega(h_{\epsilon_2}^\nu - h_{\epsilon_1}^\nu) - \frac{1}{2} h_{\epsilon_1}^\nu h_{\epsilon_2}^{-\nu} - \frac{1}{2} h_{\epsilon_2}^\nu h_{\epsilon_1}^{-\nu}, \quad (69)$$

where $h_\epsilon^\mu = 1 - 2f_\epsilon^\mu$.

Notice that the general formalism of the „nonequilibrium functional bosonization" (Eq. (66)) allows, in principle, for a nonperturbative treatment of both the elastic renormalization and the inelastic scattering if the phases $\Phi$ are kept in the exponents. (see, in particular, the calculation of the tunneling density of states in the clean LL out of equilibrium in Ref. [34]).) For our purposes in this paper, it is sufficient to expand the exponential factors to second order in $\Phi$.

The Eilenberger-type equation for the average $\langle\tilde{g}^\mu\rangle$ can now be written in the form

$$i\mu v_F \partial_x \langle\tilde{g}^\mu\rangle + \left[ i\partial_t + \frac{i}{4\tau} \langle\tilde{g}^{-\mu}\rangle, \langle\tilde{g}^\mu\rangle \right] = \hat{\mathrm{St}}_{e-e}^\mu + \hat{\mathrm{St}}_{e-b}^\mu. \quad (70)$$

The collision integrals in the right-hand side of Eq. (70) come from the averages of second order in the fluctuations of $\Phi$. There are two types of the collision integrals. One, $\hat{\mathrm{St}}_{e-b}^\mu$, comes from the second-order terms in the expansion of the phase factors $e^{\pm 2i\mu\hat{\Phi}}$ in Eq. (66). The other, $\hat{\mathrm{St}}_{e-e}^\mu$, stems from the contraction of the linear correction $\delta\tilde{g}^\mu$ with the fluctuations of $\hat{\Phi}$.

The kinetic equation for the electron distribution function is obtained by taking the Keldysh part of Eq. (70):

$$[\partial_t + \mu v_F(\partial_x + e\mathscr{E}\partial_\epsilon)]f_\epsilon^\mu = -\frac{1}{2\tau} (f_\epsilon^\mu - f_\epsilon^{-\mu})$$
$$+ \mathrm{St}_{e-e} + \sum_{b=p,g} (\mu\mathrm{St}_{e-b}^{\mathrm{el}} + \mathrm{St}_{e-b}^{\mathrm{inel}}). \quad (71)$$

The electron−boson collision integral $\mathrm{St}_{e-b}^\mu$ describes emission and absorption of the bosons of type $b = p, g$ by the fermions. Its inelastic part, which is symmetric with respect to the channel index $\mu$, reads

$$\mathrm{St}_{e-b}^{\mathrm{inel}}(\epsilon) = \pm\frac{1}{4} \sum_{\mu\nu} \int_{-\omega_0}^{\omega_0} d\omega\, \omega \mathscr{K}_{e-b}(\omega)$$

$$\times \left[ N_b^\nu(\omega)f_{\epsilon-\omega}^{-\mu}(1 - f_\epsilon^\mu) - [1 + N_b^\nu(\omega)]f_\epsilon^\mu(1 - f_{\epsilon-\omega}^{-\mu}) \right]. \quad (72)$$

The elastic (antisymmetric in $\mu$) part is given by

$$\mathrm{St}^{\mathrm{el}}_{e-b}(\epsilon) = \pm \frac{1}{4} \sum_{\mu\nu} \int\limits_{-\omega_0}^{\omega_0} d\omega\, \omega \mathscr{K}_{e-b}(\omega)$$

$$\times \mu \left[ N^{\nu}_b(\omega)(f^{\mu}_{\epsilon} - f^{\mu}_{\epsilon-\omega}) + f^{\mu}_{\epsilon} f^{-\mu}_{\epsilon-\omega} \right], \quad (73)$$

where the signs $\pm$ refer to the plasmons $(+)$ and ghosts $(-)$. In Eqs. (71)–(73), $\tau$ and $u$ are understood as renormalized down to a scale $\omega_0$, so chosen that $\Delta\epsilon \ll \omega_0 \ll \Lambda$ but $\alpha \ln(\omega_0/\Delta\epsilon) \ll 1$, where $\Delta\epsilon$ is a characteristic energy scale for the distribution function (for the quantum wire biased by a voltage $V$ it is given by $\max\{T, eV\}$). The high energy renormalization (see Sec. 3), coming from scales larger than $\omega_0$ and independent of the details of the nonequilibrium state at low energies, can thus be taken into account at the level of input parameters ($\tau$ and $u$) for the kinetic equations.

The $e-e$ collision integral $\mathrm{St}_{e-e}$ describes inelastic fermionic collisions due to the interaction via the plasmon waves:

$$\mathrm{St}_{e-e}(\epsilon) = \frac{1}{4} \sum_{\mu\nu} \int\limits_{-\omega_0}^{\omega_0} d\omega d\epsilon' \mathscr{K}_{e-e}(\omega)$$

$$\times \left[ f^{-\nu}_{\epsilon'}(1 - f^{\nu}_{\epsilon'-\omega}) f^{-\mu}_{\epsilon-\omega}(1 - f^{\mu}_{\epsilon}) \right.$$

$$\left. - f^{\nu}_{\epsilon'-\omega}(1 - f^{-\nu}_{\epsilon'}) f^{\mu}_{\epsilon}(1 - f^{-\mu}_{\epsilon-\omega}) \right]. \quad (74)$$

Note that $\mathrm{St}_{e-e}$ does not depend on the chiral indices. The collision kernels are written as

$$\mathscr{K}_{e-b}(\omega) = \pm \frac{v_F}{\omega^2 \tau} \int \frac{dq}{2\pi} \mathrm{Re} \sum_{\mu\nu} \mu\nu D^{\mu\nu}_b(\omega, q), \quad (75)$$

$$\mathscr{K}_{e-e}(\omega) = \mathscr{K}(\omega) + \mathscr{K}_{e-g}(\omega) - \mathscr{K}_{e-p}(\omega), \quad (76)$$

where

$$\mathscr{K}(\omega) = -\frac{1}{\pi\omega} \int \frac{dq}{2\pi} \sum_{\mu\nu} \mathrm{Re} D^{\mu\nu}_g(\omega, q) \mathrm{Im} V^{\nu\mu}_R(\omega, q) \quad (77)$$

and the explicit form of the propagators $D^{\mu\nu}_g$ and $V^{\mu\nu}$ can be found in Ref. [12] (see Eqs. (A16), (A17), (A21)−(A23) there; note that $D^{\mu\nu}_g$ corresponds to $v_F D_{\mu\nu}$ in Eqs. (A16), (A17)).

For the electron−boson terms we obtain in the ballistic limit of energy transfers $\omega$ larger than $\tau^{-1}$ the simple expressions

$$\mathscr{K}_{e-p}(\omega) = \frac{2K}{\omega^2 \tau}, \quad \mathscr{K}_{e-g}(\omega) = \frac{2}{\omega^2 \tau}, \quad \omega \gg \tau^{-1}. \quad (78)$$

The asymptotic behavior of $\mathscr{K}(\omega)$ in three parametrically different ranges of $\omega$ in the limit of weak interaction $\alpha \ll 1$ is as follows:

$$\mathscr{K}(\omega) = \begin{cases} 2\alpha^2/\omega^2\tau, & \tau^{-1} \ll \omega \ll \alpha T_1, \\ 8\alpha^4\tau, & \alpha T_1 \ll \omega \ll T_1, \\ 2/\omega^2\tau, & \omega \gg T_1, \end{cases} \quad (79)$$

where $T_1 \sim 1/\alpha^2\tau$ is the characteristic temperature below which the localization effects are strong (see the discussion
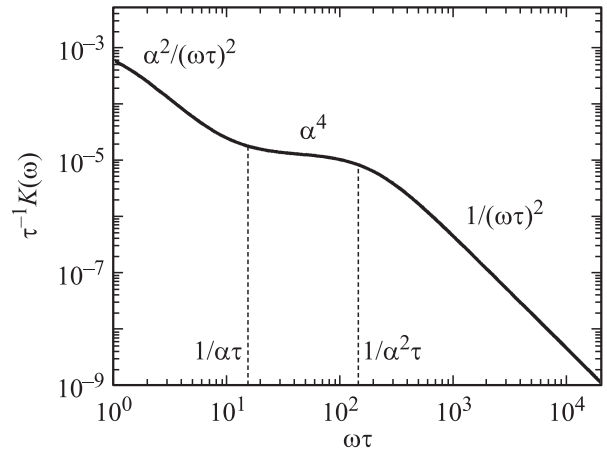


**Figure 2.** Frequency dependence of the collision kernel $\mathscr{K}(\omega)$ ([Eq. (77)]) for $\alpha = 0.05$. Three different types of scaling behavior of the dimensionless product $\tau^{-1}\mathscr{K}(\omega)$ are indicated, as well as the characteristic values of $\omega$.

at the end of Sec. 4). The log-log plot of $\mathscr{K}(\omega)$ in the whole range of $\omega$ for a particular value of $\alpha$ (taken very small for the purpose of illustration) is shown in Fig. 2.

An important feature of $\mathscr{K}(\omega)$ in Eq. (79) is its nonperturbative behavior with respect to $\alpha$ at $\omega \gg \alpha T_1 \sim 1/\alpha\tau$. In particular, for $\omega \gg T_1$ the $e-e$ collision kernel does not at all depend on the $e-e$ interaction strength. The origin of the nonperturbative dependence on $\alpha$, despite $\alpha \ll 1$ being small, is related to the analytical structure of $\mathscr{K}(\omega)$ in Eq. (77). Specifically, the integrand of $\mathscr{K}(\omega)$ contains eight poles, which in the limit of large $\omega$ are only slightly „damped" by disorder: $q \approx \pm\omega(1 \pm i/2\omega\tau)/v_F$ and $q \approx \pm\omega(1 \pm i/2\omega\tau)/u$. As a result, the contour of integration in the plane of $q$ is squeezed between two close poles $(u \to v_F$ for $\alpha \to 0)$, one of which is in the upper-half plane and the other in the lower one. At $\omega \gg T_1$ we thus have $\mathscr{K}(\omega) \propto \alpha^2/|u - v_F|$ and $\alpha$ drops out altogether.

The kinetic equations for the bosonic distribution functions $N^{\mu}_{p,g}$ follow from Eqs. (59), (61), (62):

$$(\partial_t + \mu u_b \partial_x) N^{\mu}_b(\omega) = -\frac{1}{\tau} N^{\mu}_b(\omega) + \mathrm{St}_{b-e}(\omega), \quad (80)$$

where

$$\mathrm{St}_{b-e}(\omega) = \frac{1}{2\omega\tau} \sum_{\mu} \int d\epsilon\, f^{\mu}_{\epsilon}(1 - f^{-\mu}_{\epsilon-\omega}) \quad (81)$$

describes the creation of the boson from an electron−hole pair, where the electron and hole move in the opposite directions.

The role of the ghost modes can be further elucidated if one considers the energy conservation law. The electronic and bosonic energy densities $\rho^{\epsilon}_e$, $\rho^{\epsilon}_b$ and the energy current

densities $j_e^\epsilon$, $j_b^\epsilon$ are given by

$$\rho_e^\epsilon = \frac{1}{2\pi v_F} \int_{-\infty}^{\infty} d\epsilon\, \epsilon (f_\epsilon^+ + f_\epsilon^-), \qquad (82)$$

$$j_e^\epsilon = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\epsilon\, \epsilon (f_\epsilon^+ - f_\epsilon^-), \qquad (83)$$

$$\rho_b^\epsilon = \frac{1}{2\pi u_b} \int_{0}^{\infty} d\omega\, \omega [N_b^+(\omega) + N_b^-(\omega)], \qquad (84)$$

$$j_b^\epsilon = \frac{1}{2\pi} \int_{0}^{\infty} d\omega\, \omega [N_b^+(\omega) - N_b^-(\omega)]. \qquad (85)$$

The prefactors in Eqs. (82) and (84) represent the density of states for electrons and bosons, respectively. Then the kinetic equations (71) and (80) assure the conservation law

$$\partial_t(\rho_e^\epsilon + \rho_p^\epsilon - \rho_g^\epsilon) + \partial_x(j_e^\epsilon + j_p^\epsilon - j_g^\epsilon) = j_e \mathscr{E}, \qquad (86)$$

where in the right-hand side $\mathscr{E}$ is the applied electric field and $j_e$ is the induced current of charge. The kinetic energy without any interaction is determined by the energy of the bare electrons. In the presence of Coulomb interaction, the plasmon energy is given by a sum of the $e-e$ interaction energy and the kinetic energy of the bare electron-hole pairs, the latter being the ghost energy by construction. The total energy is thus given by the sum of the plasmon and electron systems with a subtraction of the energy of the ghosts.

We now turn to the rate of energy relaxation $\tau_E^{-1}$ in the limit of weak nonequilibrium by linearizing the kinetic equations (71) and (81). One sees that at large energy transfers the inelastic $e-e$ scattering dominates over electron–boson collisions: $\mathscr{K}(\omega) \gg \mathscr{K}_{e-g}(\omega) - \mathscr{K}_{e-b}(\omega)$ for $\omega \gg 1/\alpha^{3/2}\tau$. Assuming that the large $\omega$ give the main contribution to the energy relaxation in the limit $T \gg T_1$, where the localization effects can be neglected (see Sec. 4), the equilibration rate at which the fermionic system relaxes to a locally equilibrium Fermi distribution is estimated as

$$\frac{1}{\tau_E(T)} \sim \frac{1}{T} \int_{0}^{T} d\omega\, \omega^2 \mathscr{K}(\omega) \sim T^2 \mathscr{K}(T) \sim \frac{1}{\tau}. \qquad (87)$$

Notice that the characteristic $\omega$ in Eq. (87) is of order $T$, which justifies the use of $\mathscr{K}(\omega)$ only. On the other hand, this makes it impossible to describe the equilibration in terms of the much simpler Fokker–Planck equation in the energy space. Remarkably, the equilibration rate (87) does not depend on the strength of interaction and is given by the elastic scattering rate. The interaction constant $\alpha$ enters only through the condition $T \gg T_1$. The equilibration rate turns out to be much smaller than the (clean) $e-e$ collision rate (22) and also much smaller than the dephasing rate (30):

$$\tau_E^{-1} \ll \tau_\phi^{-1} \ll \tau_{ee}^{-1}. \qquad (88)$$

It is also worth emphasizing that the characteristic energy transfers are parametrically different in these three types of relaxation processes.

## 6. Summary

In this paper, we have studied the relaxation properties of interacting spinless electrons in a disordered quantum wire within the Luttinger-liquid model. We first review the basic concepts in the disordered Luttinger liquid at equilibrium, including the elastic renormalization, dephasing, and interference-induced localization. We have introduced the general framework for describing the relaxation processes in the strongly correlated (non-Fermi) electron system at nonequilibrium. Our main result is the coupled set of the kinetic equations for the fermionic (71) and bosonic (80) distribution functions. The peculiarity of the Luttinger liquid model is that the electron–electron scattering rate (the inverse lifetime of the fermionic excitations) is finite at nonzero temperature but the energy exchange is exactly zero in the clean limit. The energy relaxation occurs only due to the scattering off disorder. We have calculated the energy equilibration rate that turns out to be independent of the strength of electron–electron interaction at sufficiently high temperature, when the Anderson localization effects are suppressed, and equal to the rate of elastic scattering off disorder.

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